Locality and bounding-box quality of two-dimensional space-filling curves

Herman Haverkort   Freek van Walderveen

European Space-filling Agency
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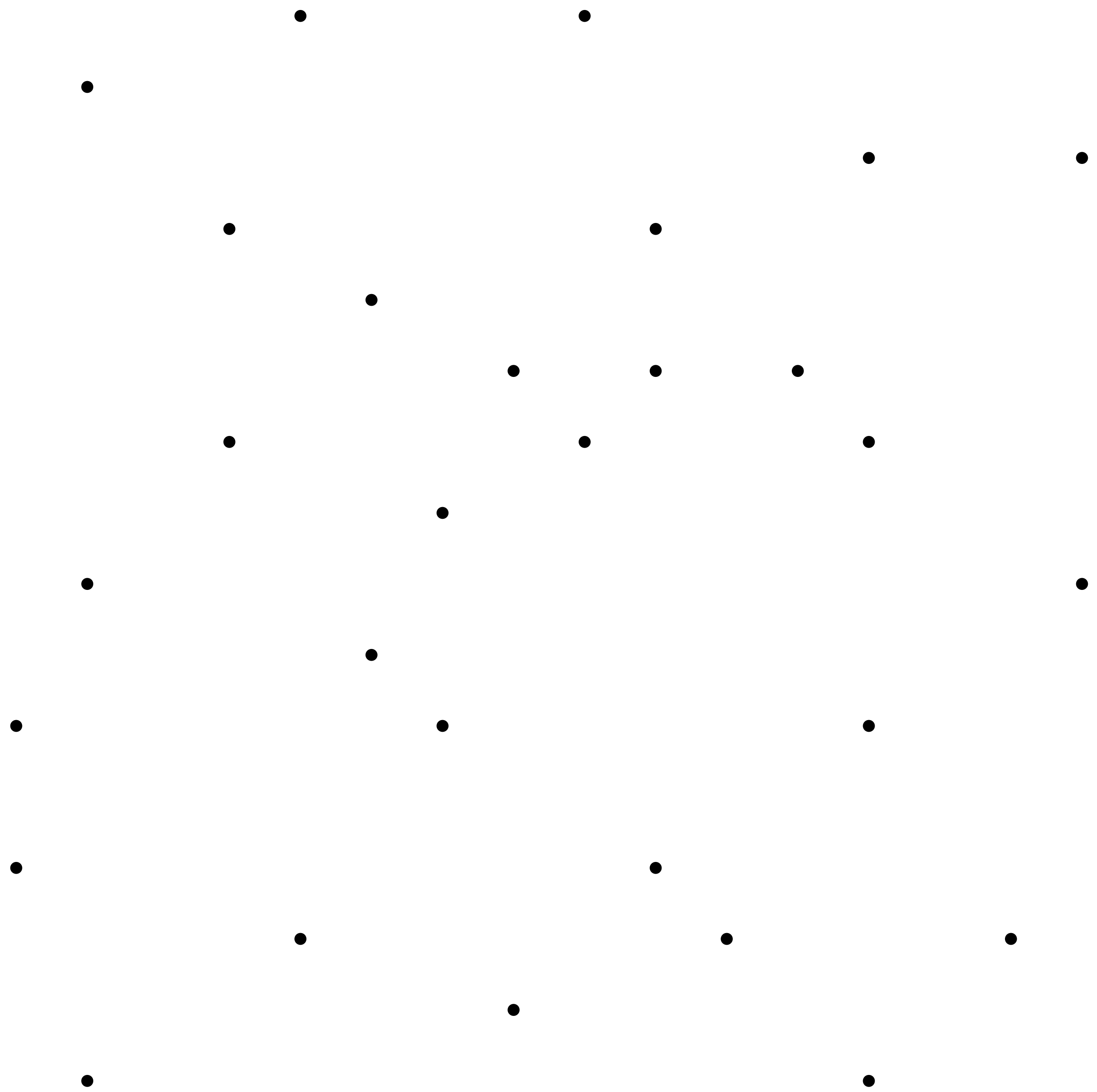
Ordering and grouping points



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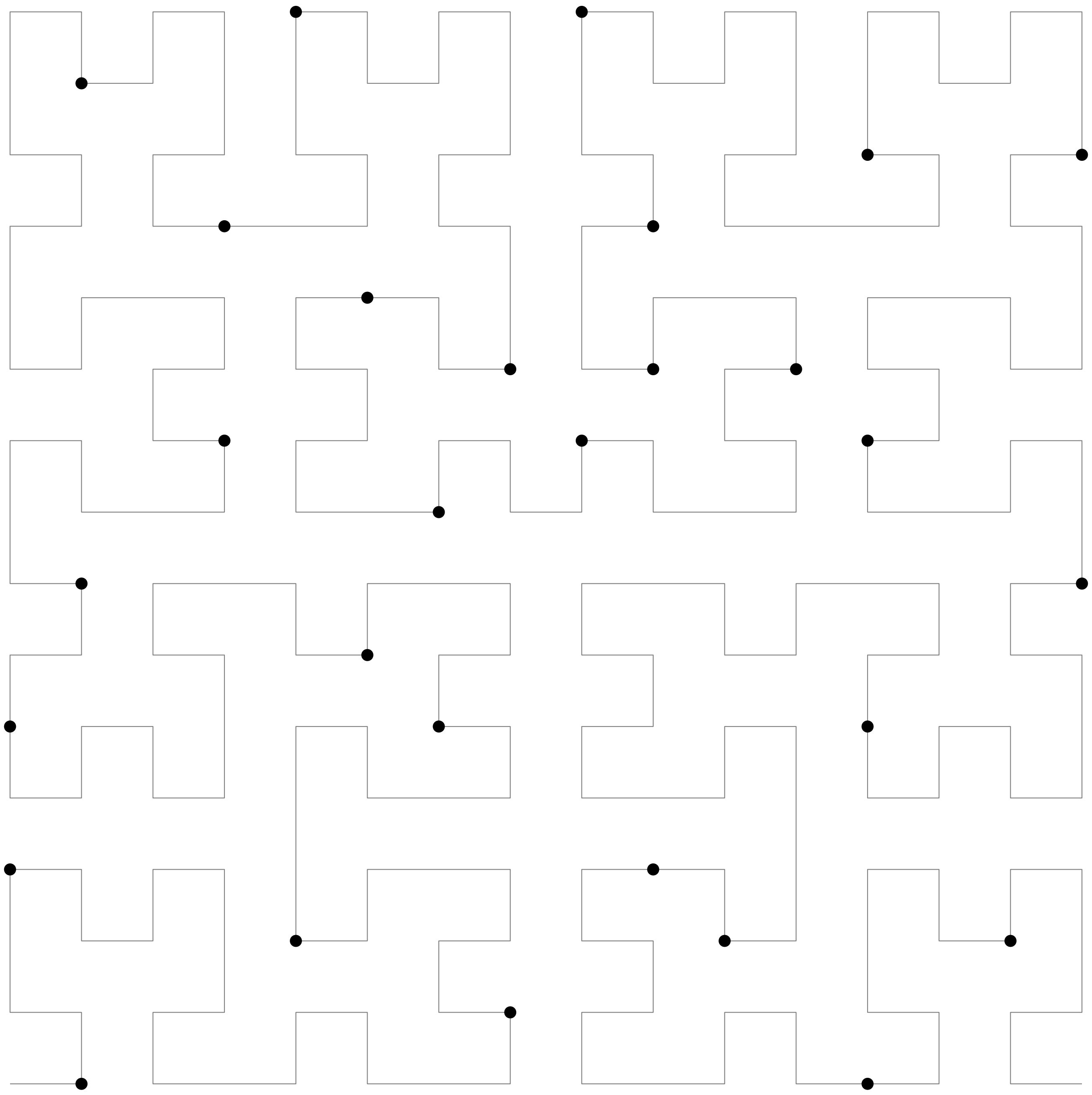

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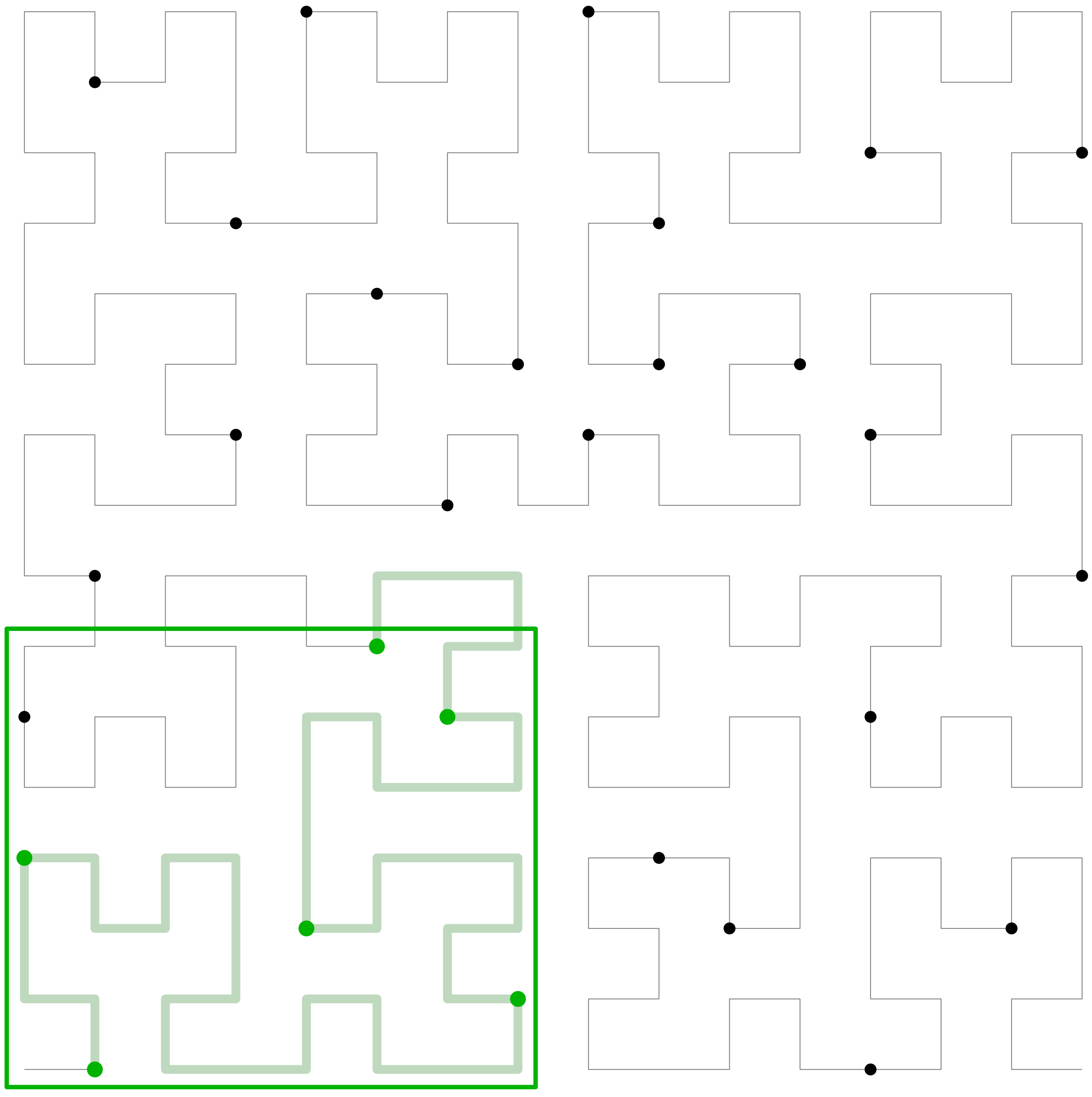

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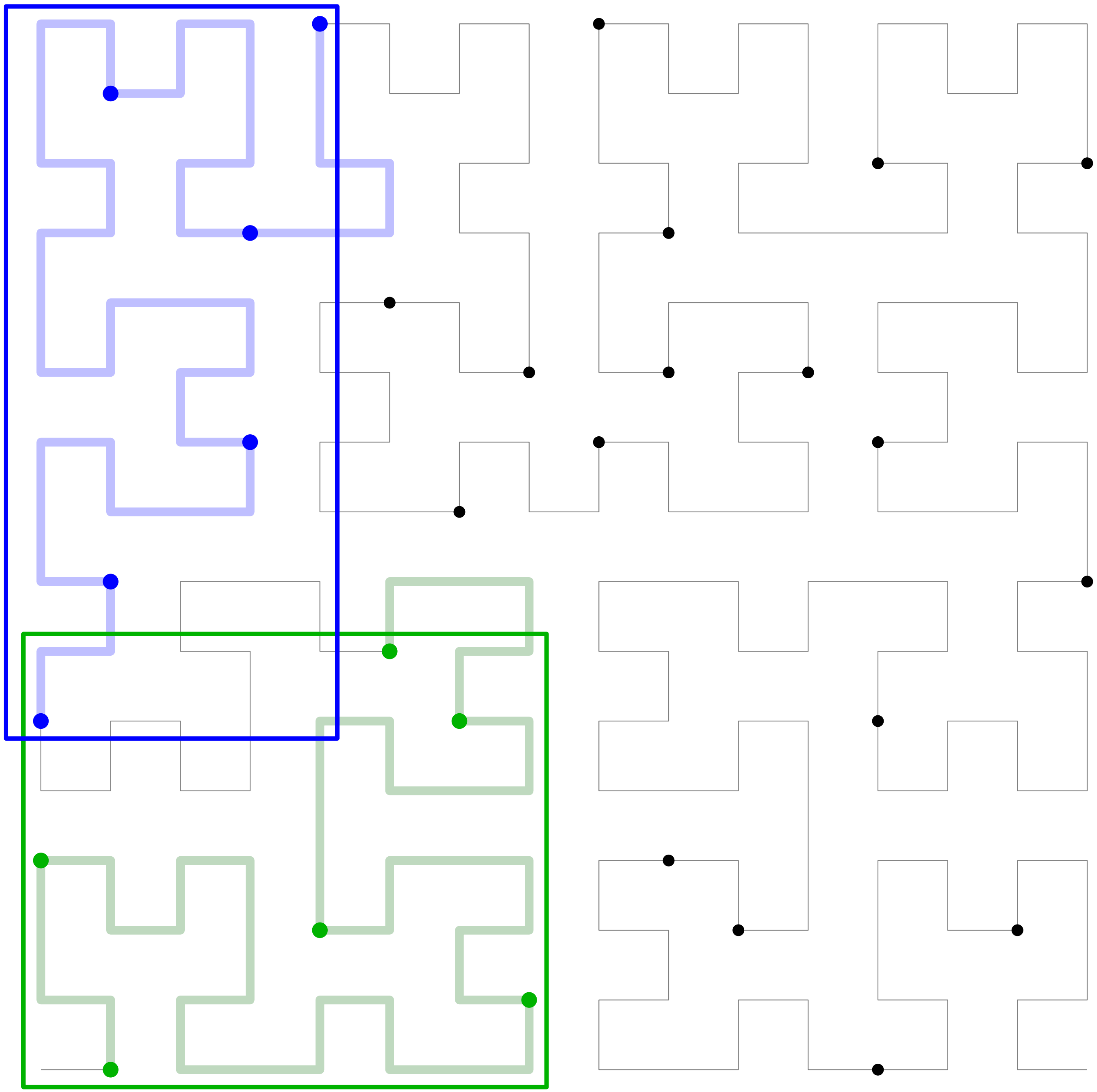

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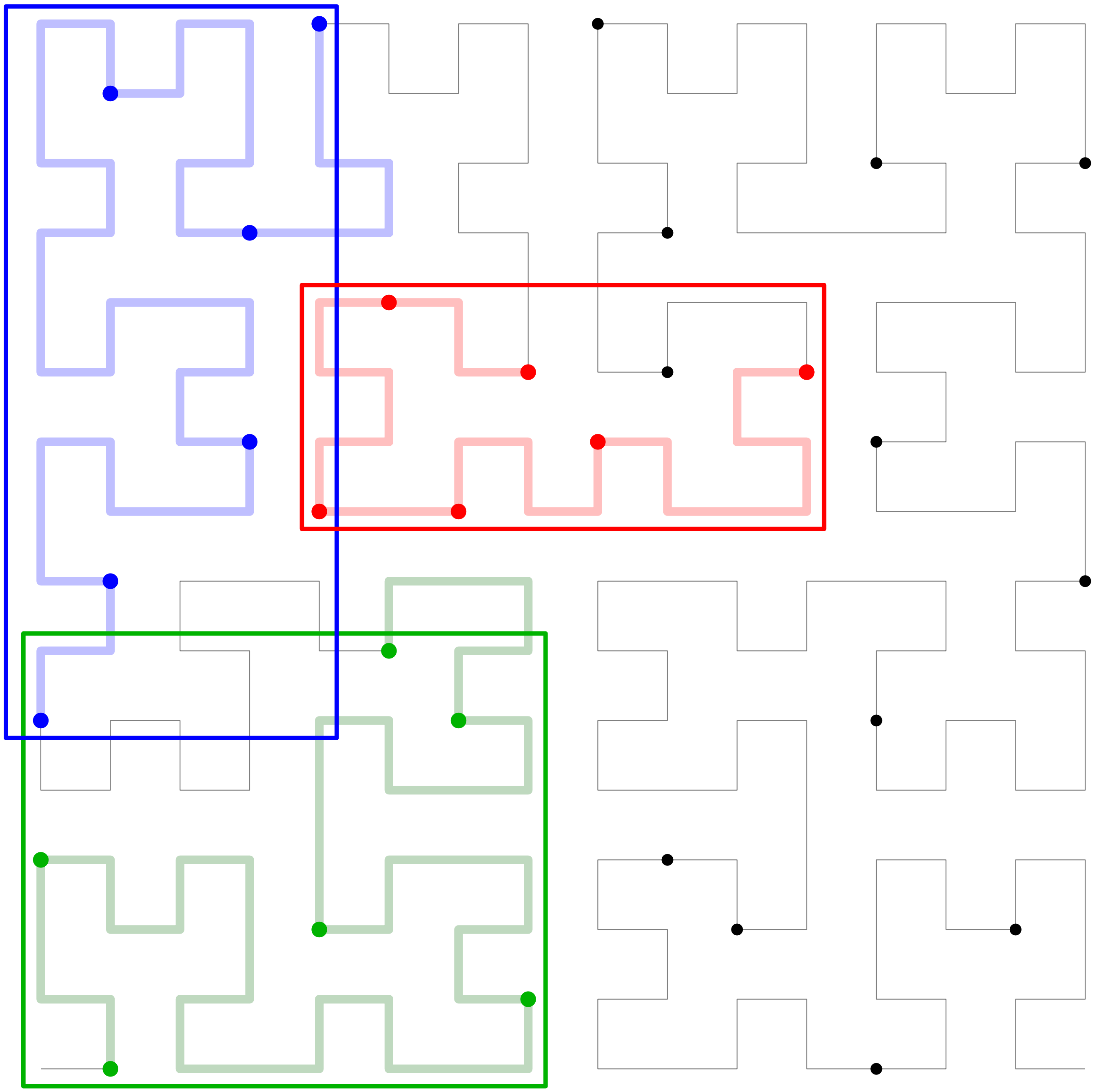

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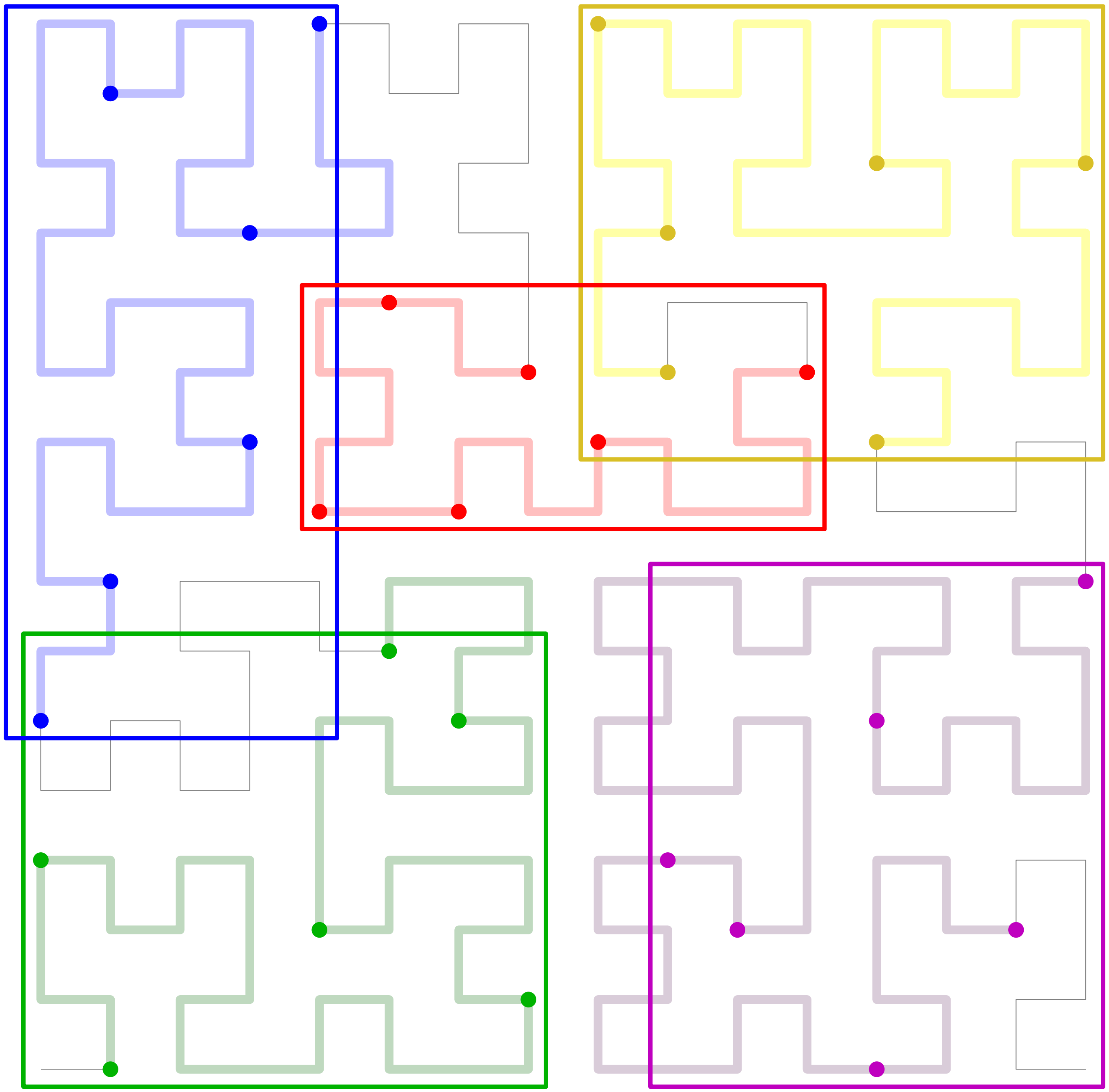

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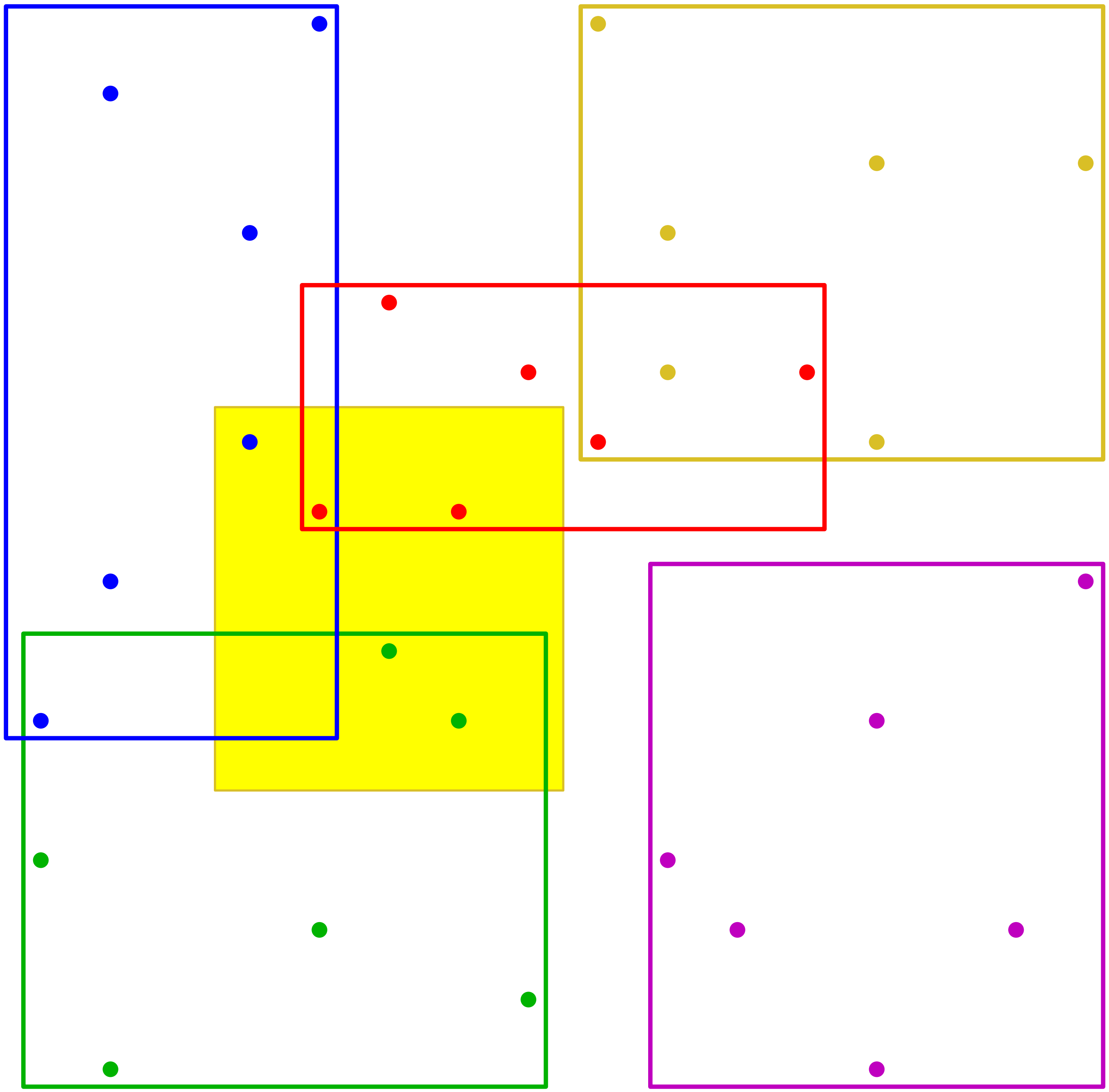

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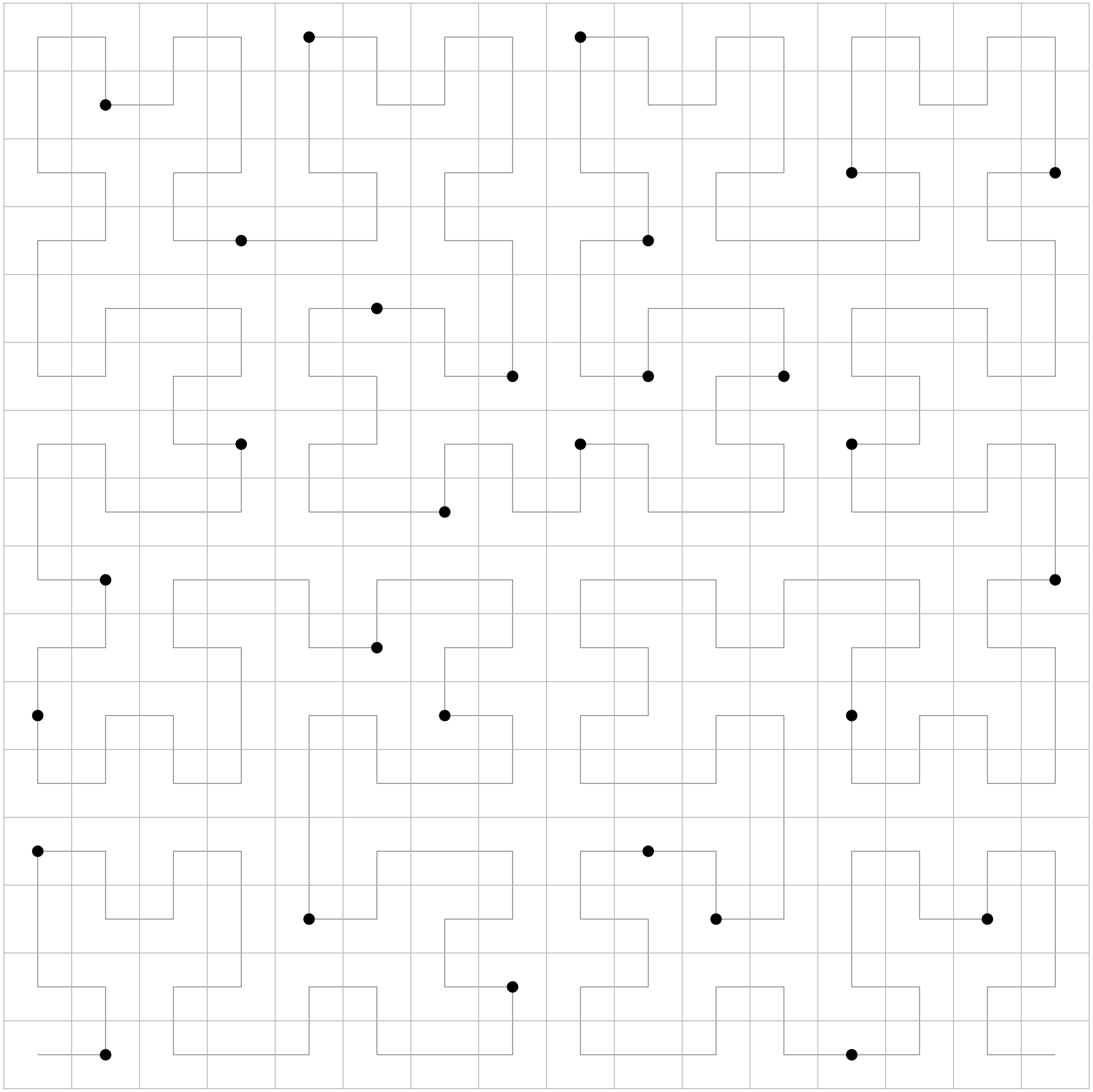

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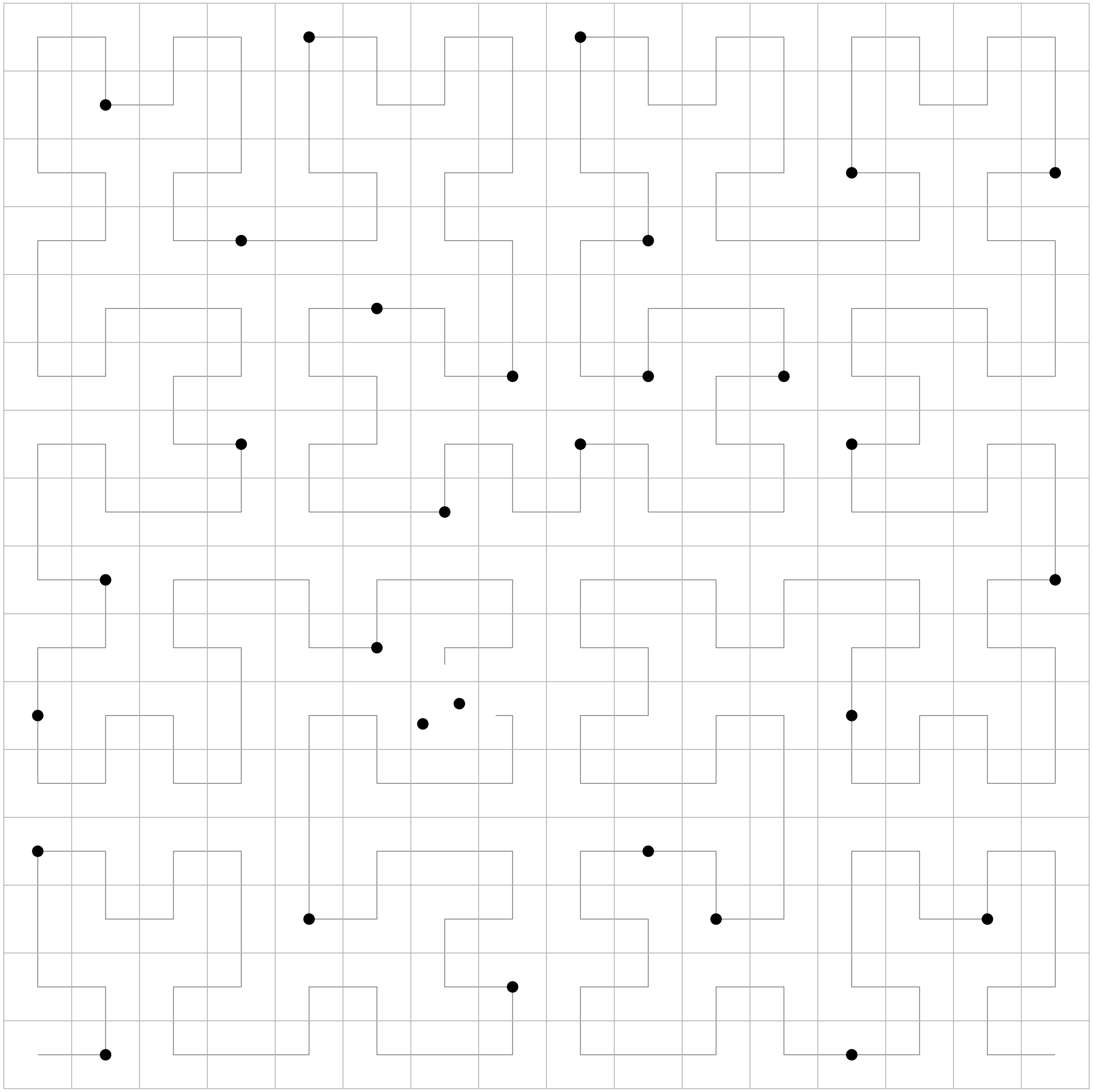

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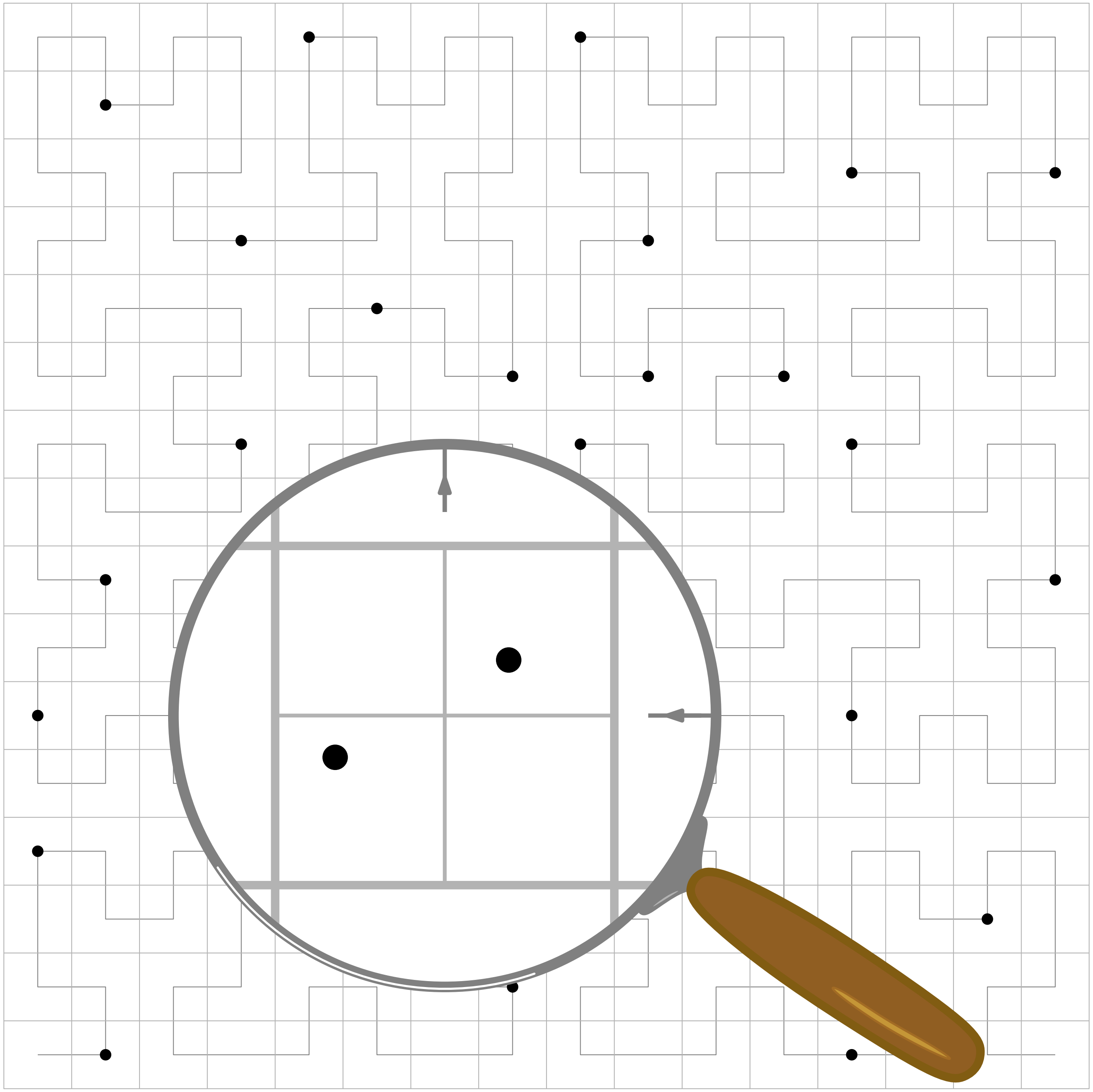

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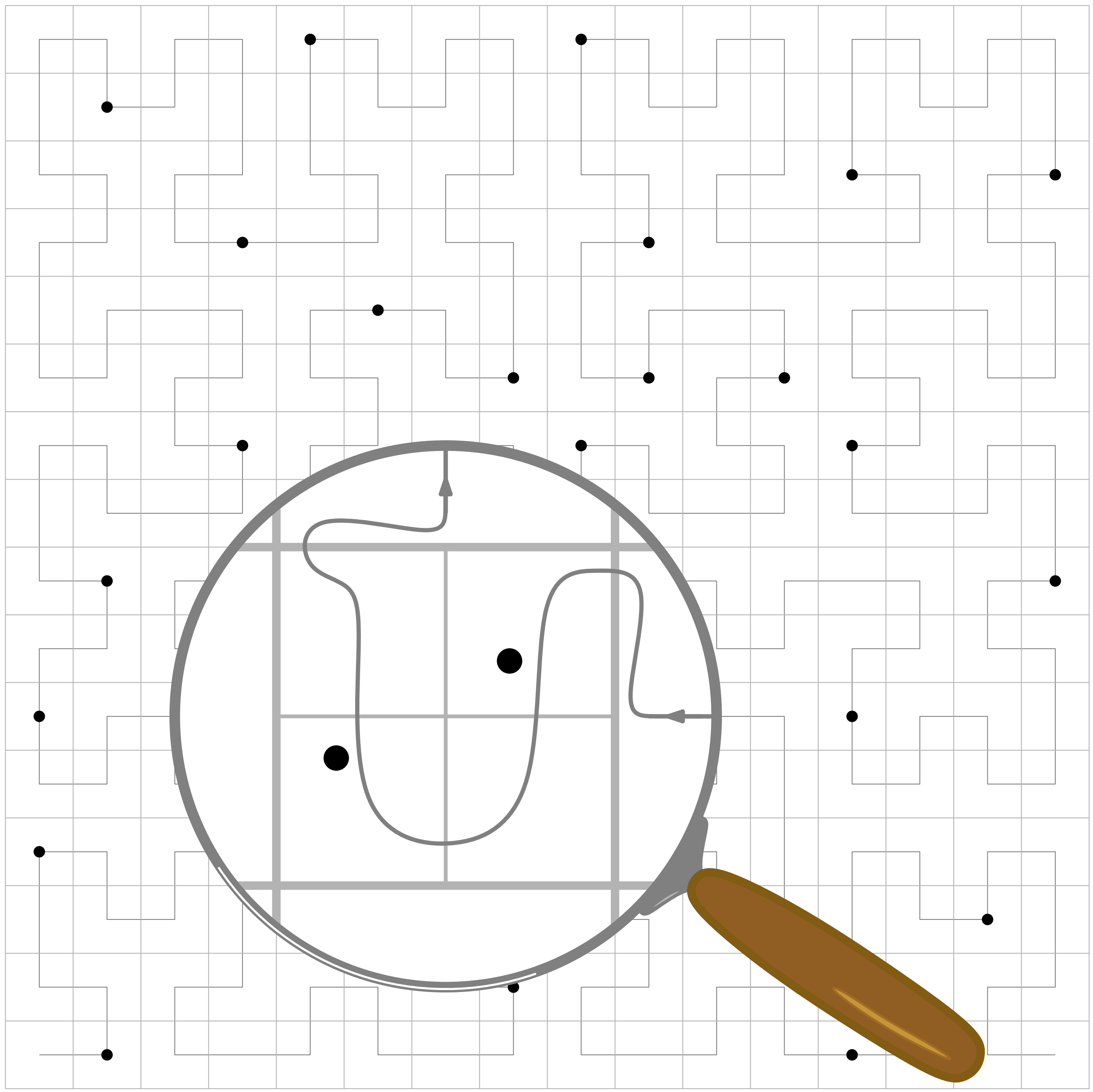
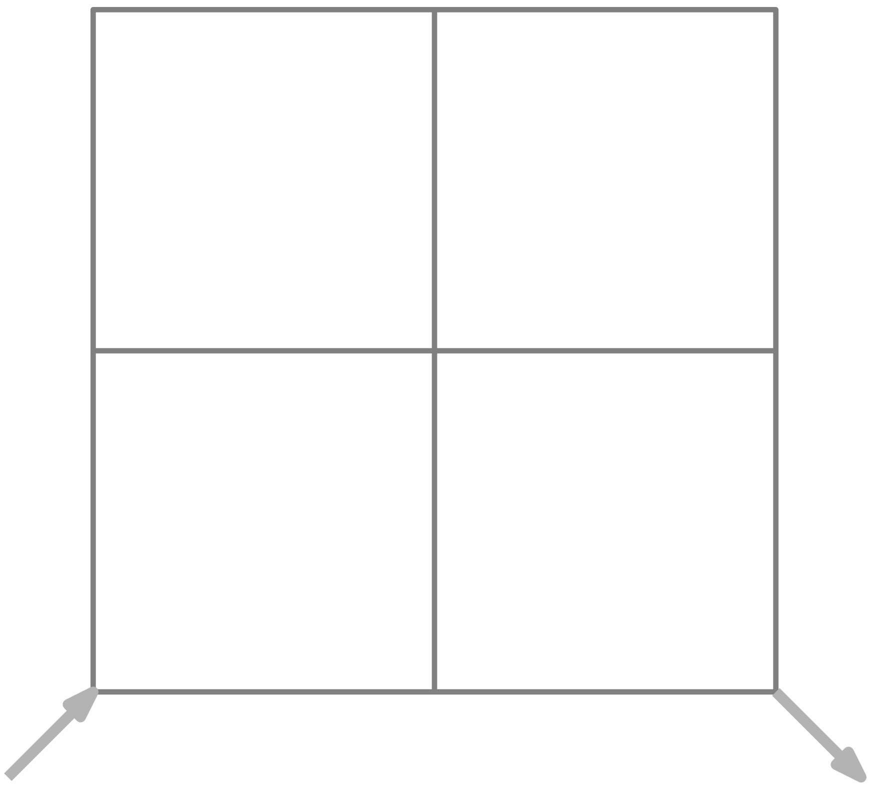


Defining a space-filling curve



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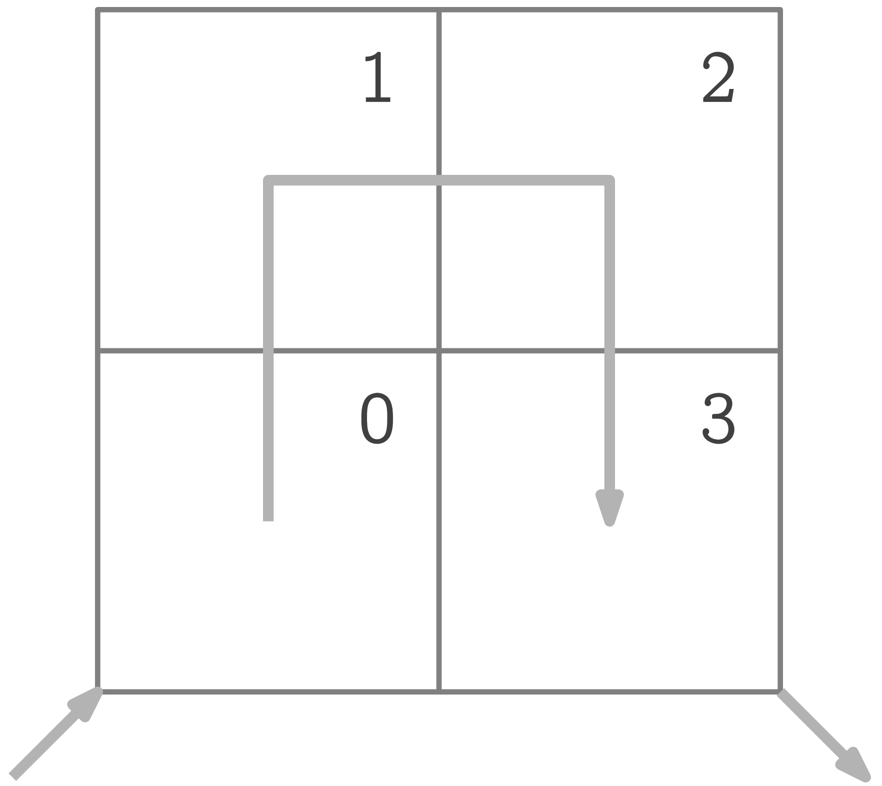

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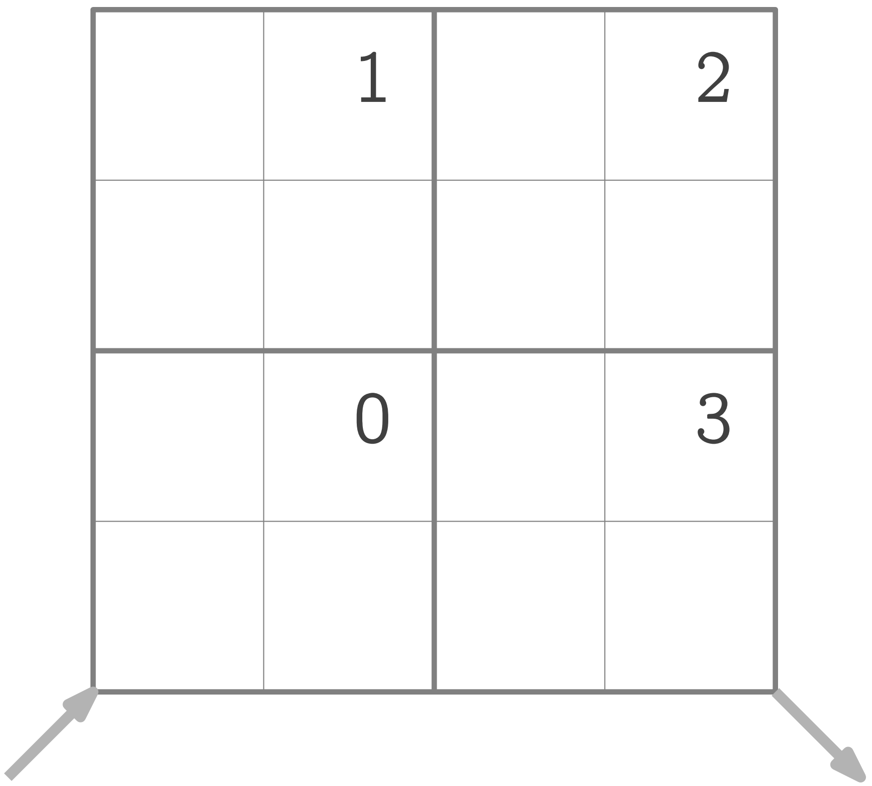

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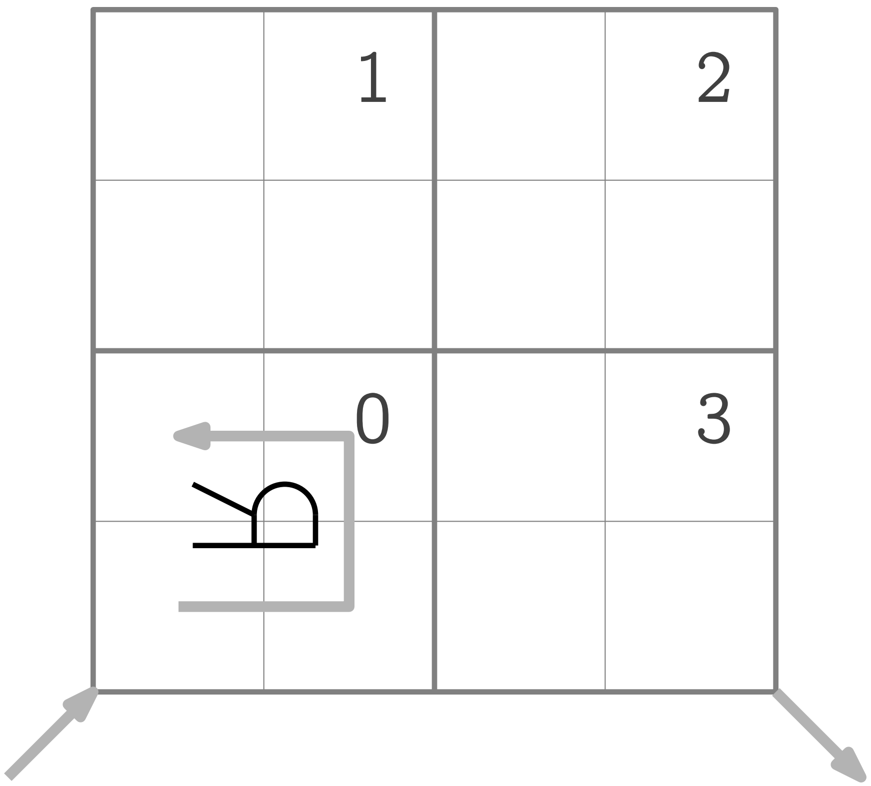

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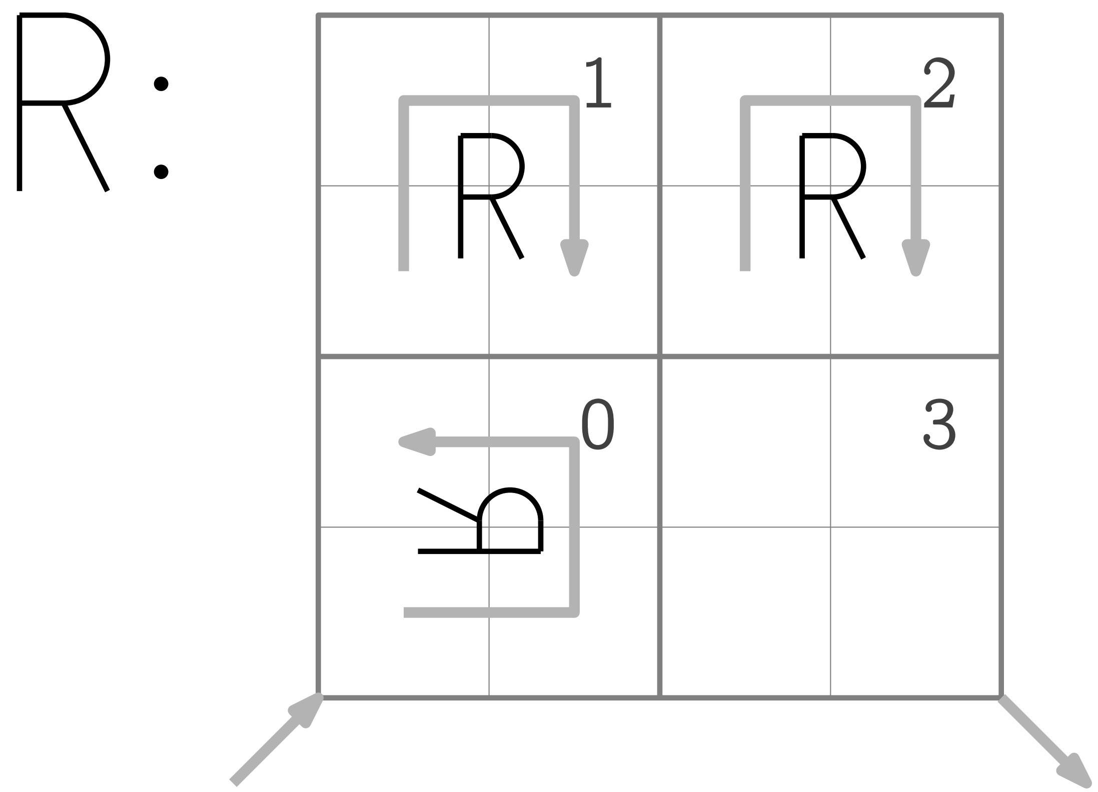

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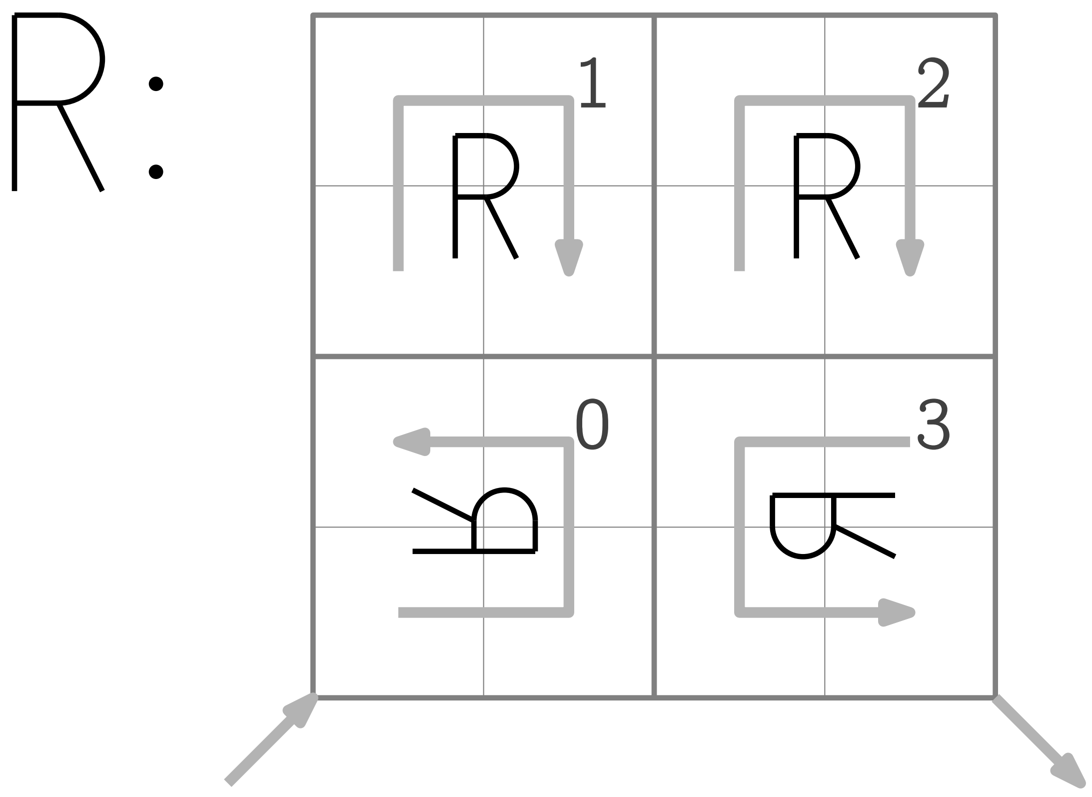

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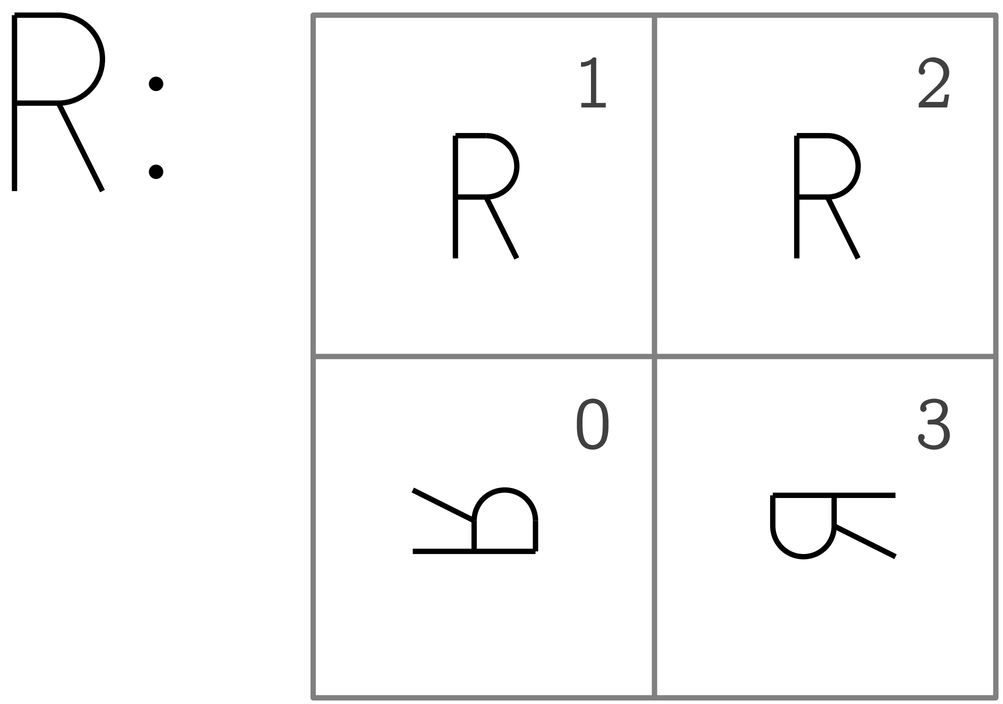
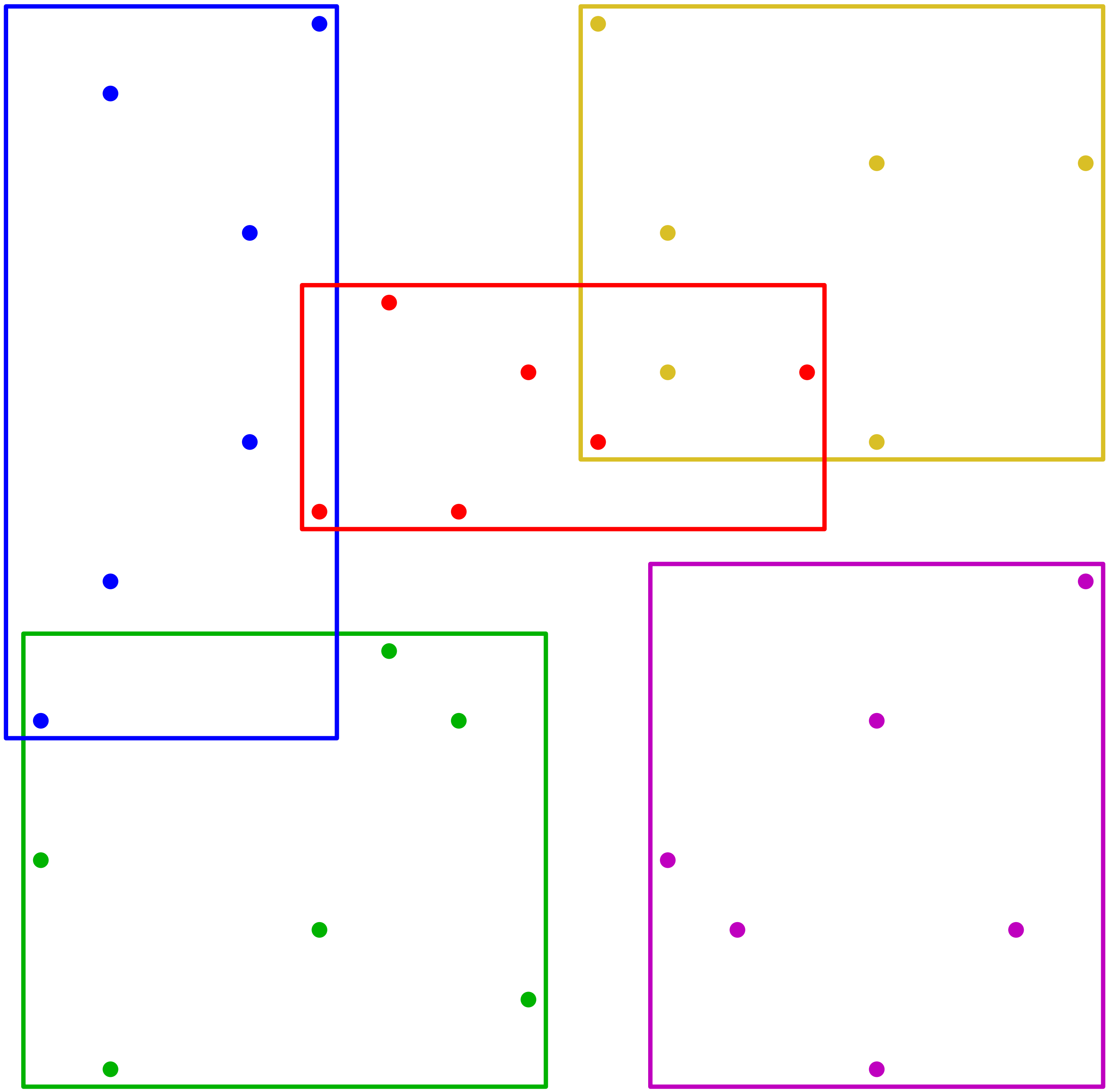


Hilbert's space-filling curve

Quality measures



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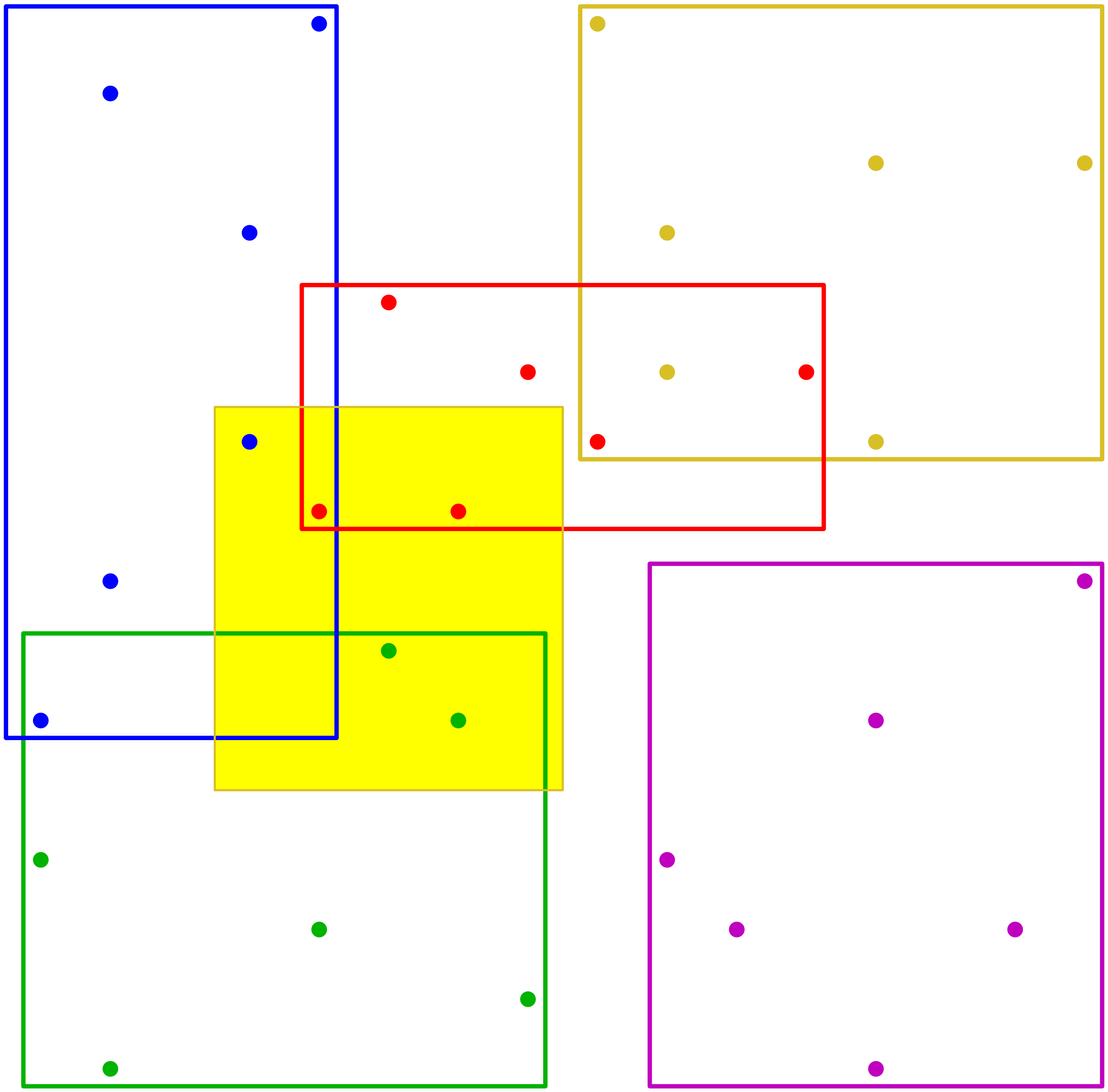
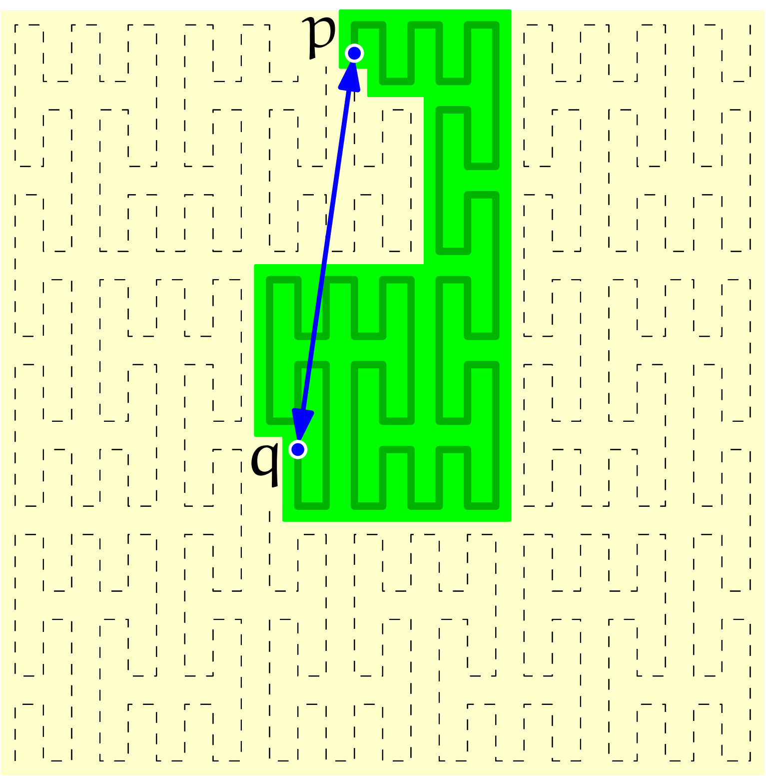


Quality measures

Literature: [Mandelbrot 1983], [Chochia et al. 1995], [Gotsman & Lindenbaum 1996], [Alber & Niedermeier 2000], [Niedermeier et al. 2002], [Bauman 2006], etc.

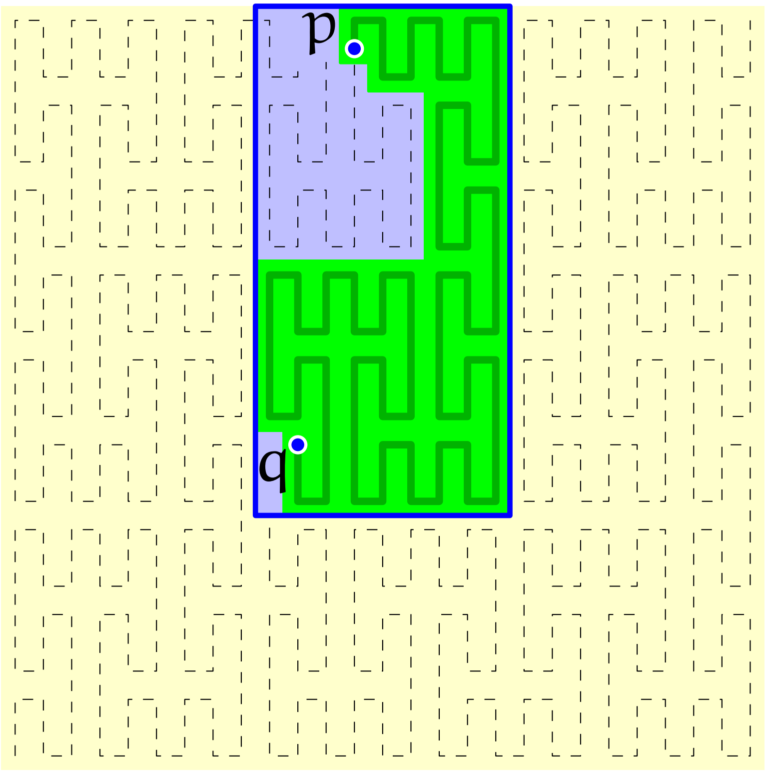
$$\text{worst-case locality} := \max_{p,q \in \text{unit}\square} \frac{\text{squared distance between } p \text{ and } q}{\text{area filled by curve between } p \text{ and } q}$$



For a curve section of fixed size, how far can the endpoints be apart?

Quality measures

$$\text{worst-case bbox area} := \max_{p,q \in \text{unit}\square} \frac{\text{bbox area of } C(p,q)}{\text{area filled by } C(p,q)}$$



For a curve section of fixed size, how big can the bounding box be?

Worst-case bounding-box area $\geqslant 2$

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assumptions

- Grid is rectangular, regular and recursively refinable.

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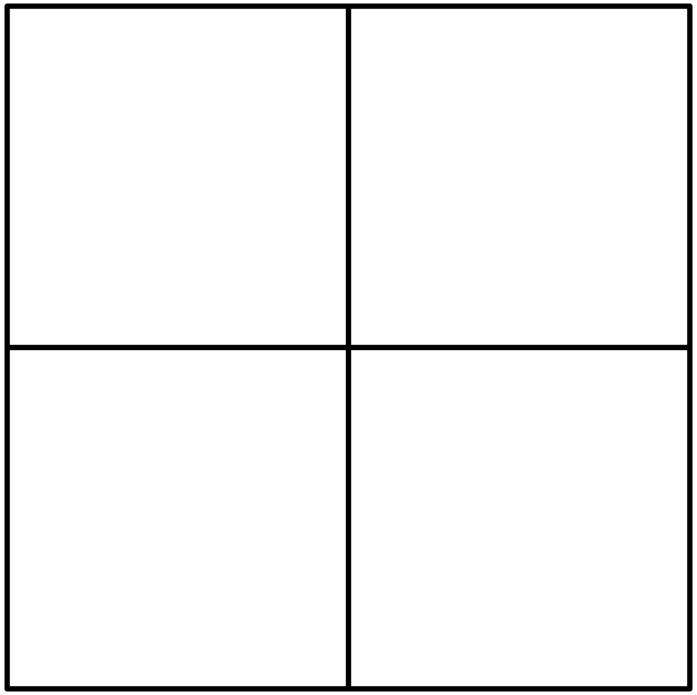

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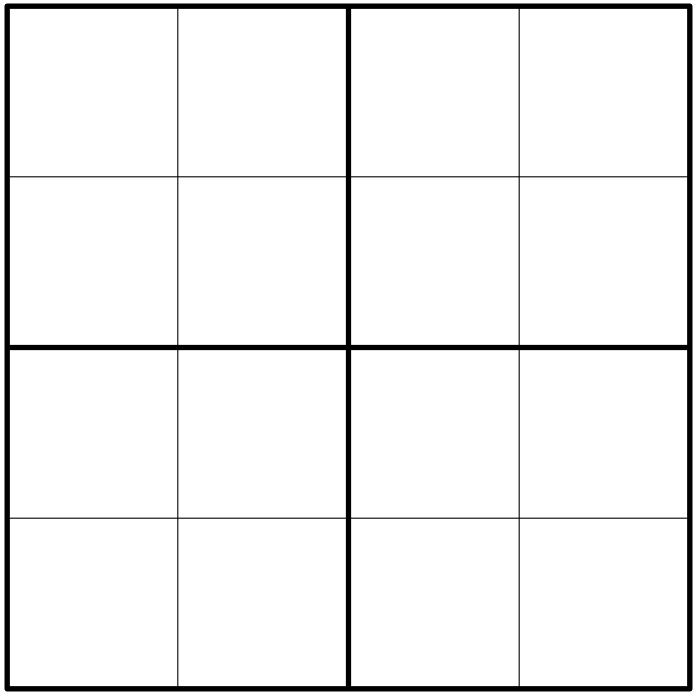

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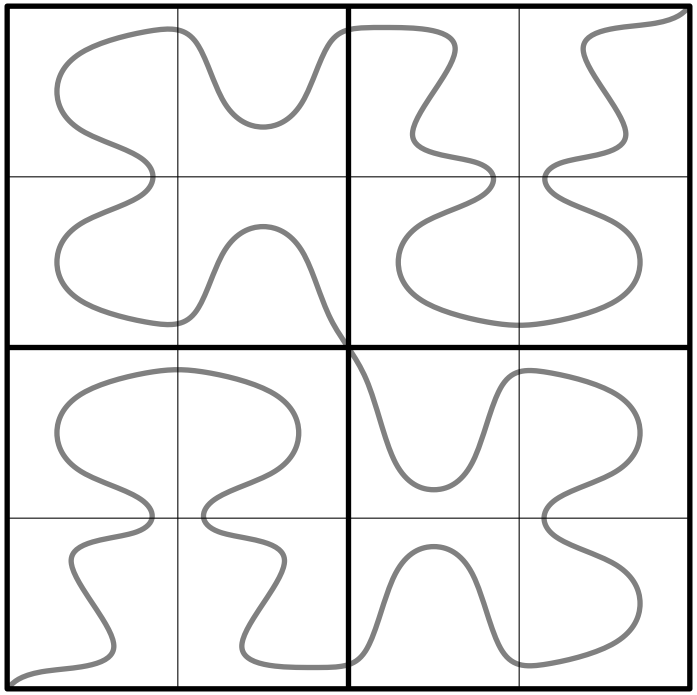

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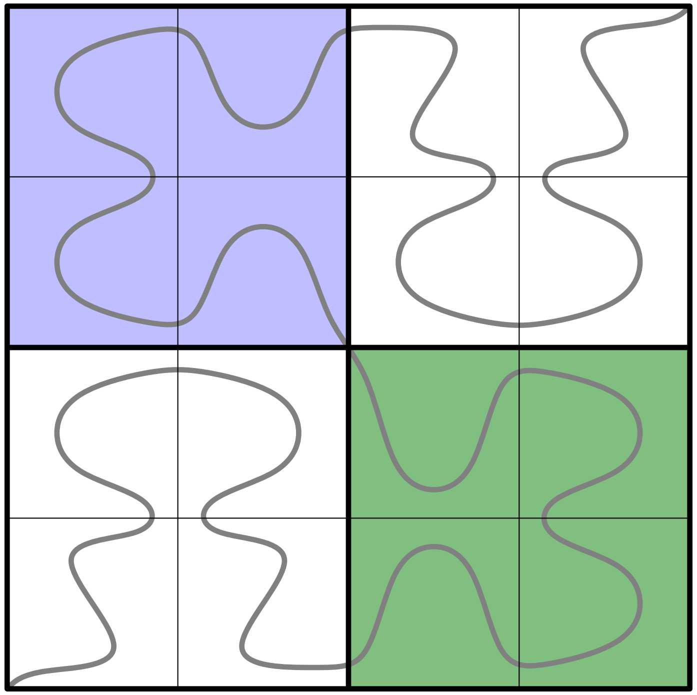

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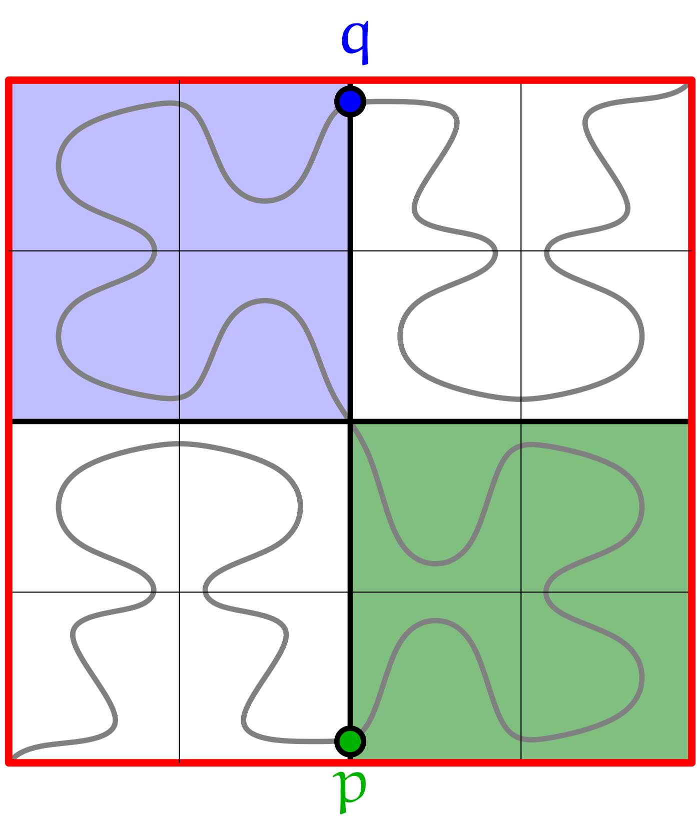

$$\mathrm{WBA} \geqslant \frac{|\mathrm{bbox}(C(p,q))|}{|C(p,q)|} = \frac{1}{1/4 + 1/4} = 2$$

Worst-case bounding-box area $\geqslant 2$

- Grid is rectangular, regular and recursively refinable.
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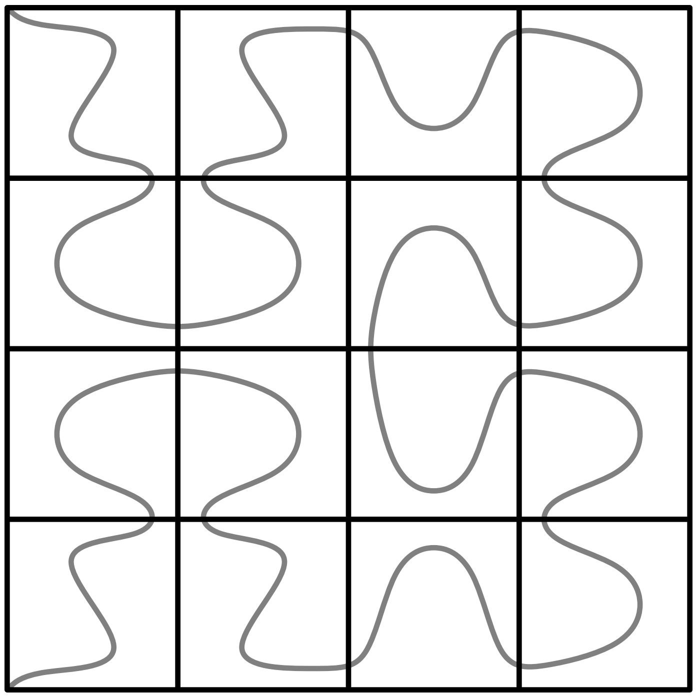
- Case: curve is *edge-connected*.



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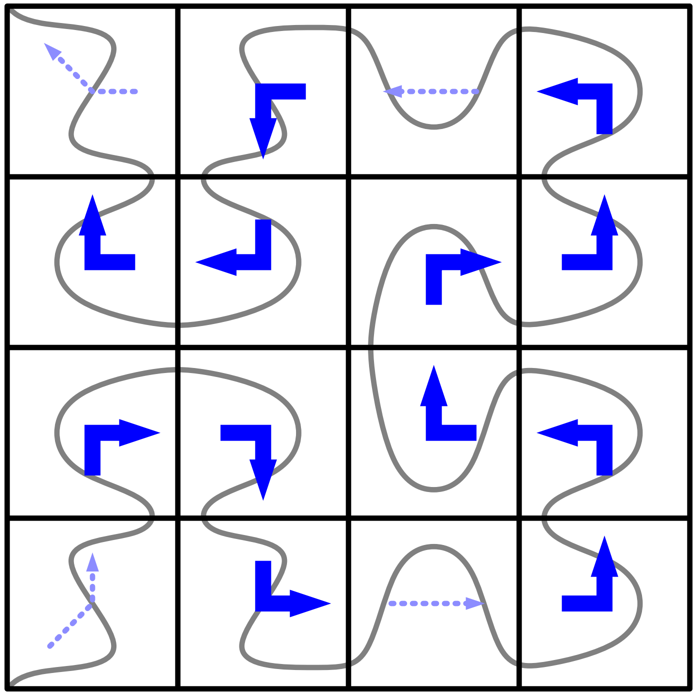

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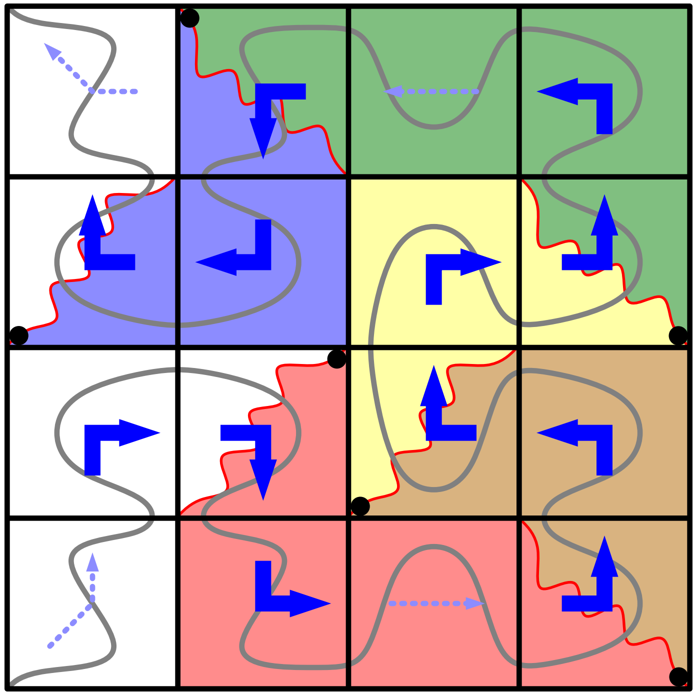

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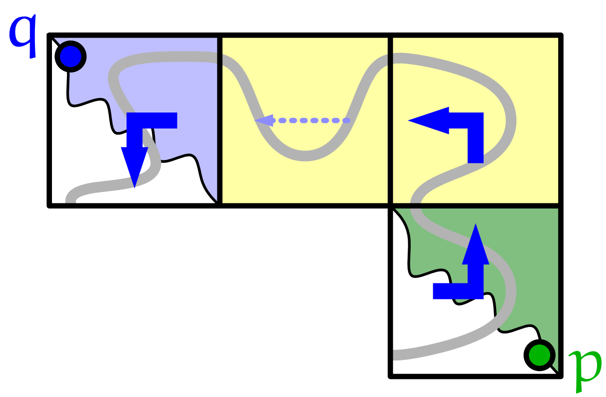

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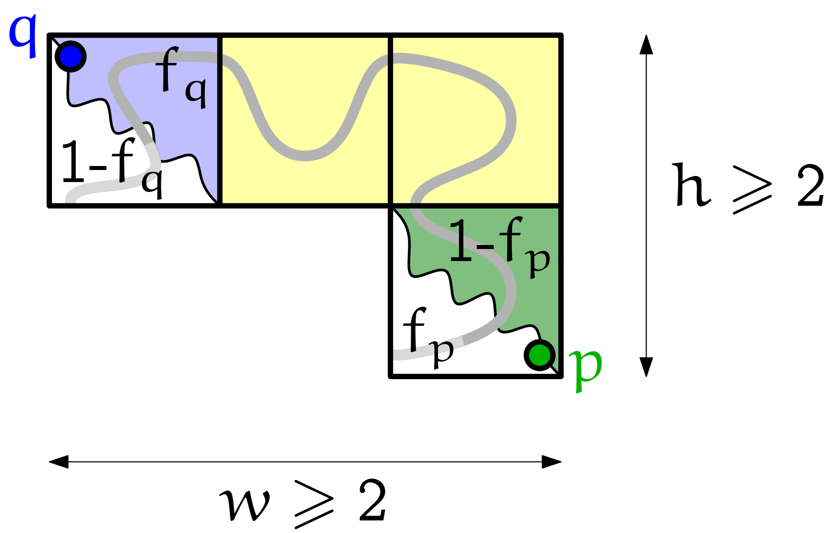

$$\mathrm{WBA} \geqslant \frac{|\mathrm{bbox}(C(p,q))|}{|C(p,q)|}$$

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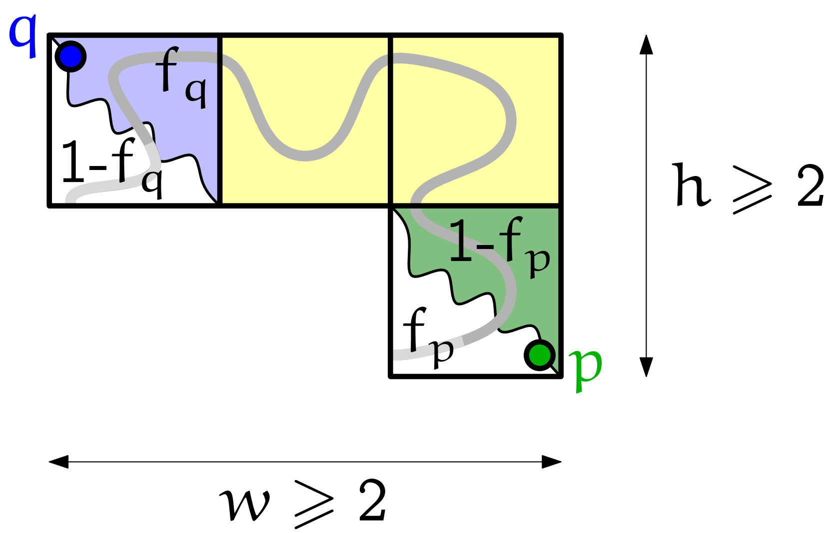

$$\mathrm{WBA} \geqslant \frac{|\mathrm{bbox}(C(p,q))|}{|C(p,q)|} = \frac{w \cdot h}{w+h-1-f_p-(1-f_q)}$$

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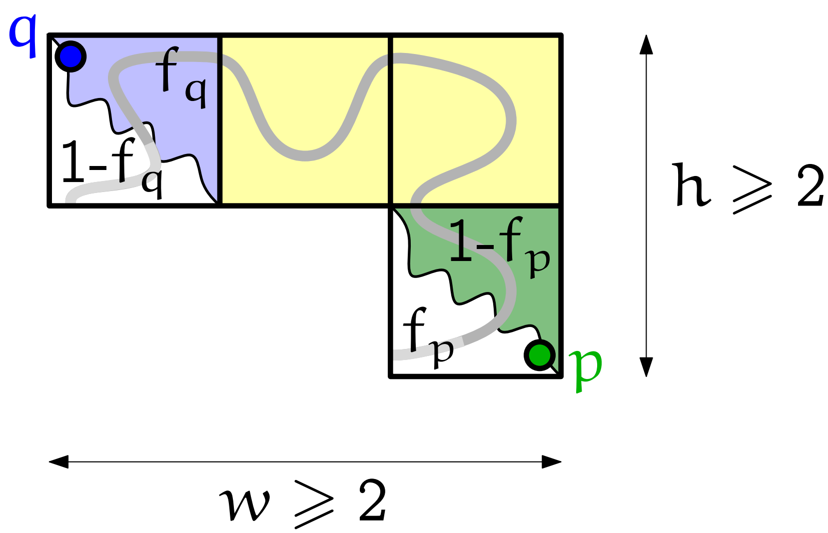

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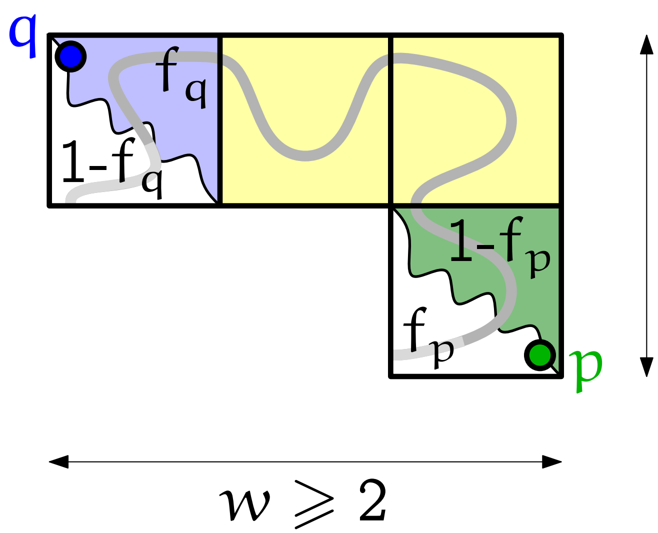

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\text{WBA} &\geqslant \frac{|\text{bbox}(C(p,q))|}{|C(p,q)|} \\
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\end{aligned}$$

$$\begin{aligned}
f_q - f_p &\geqslant \left(\frac{2}{\text{WBA}} - 1\right)(w + h - 2) \\
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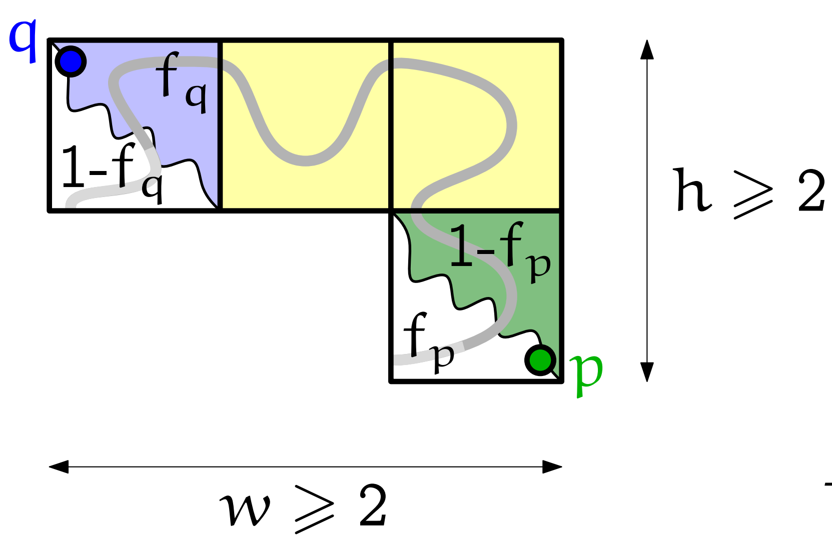

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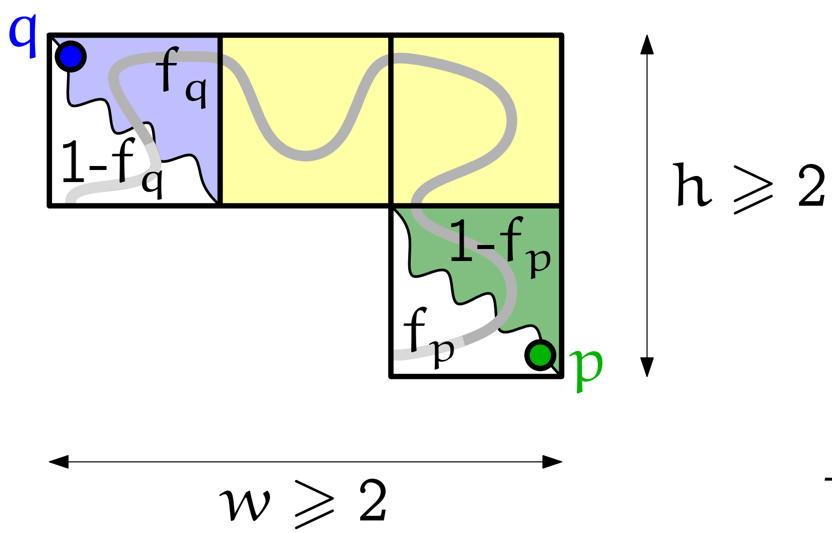

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$$\mathrm{WBA} < 2 \Rightarrow \forall p, q : f_q - f_p > 0$$

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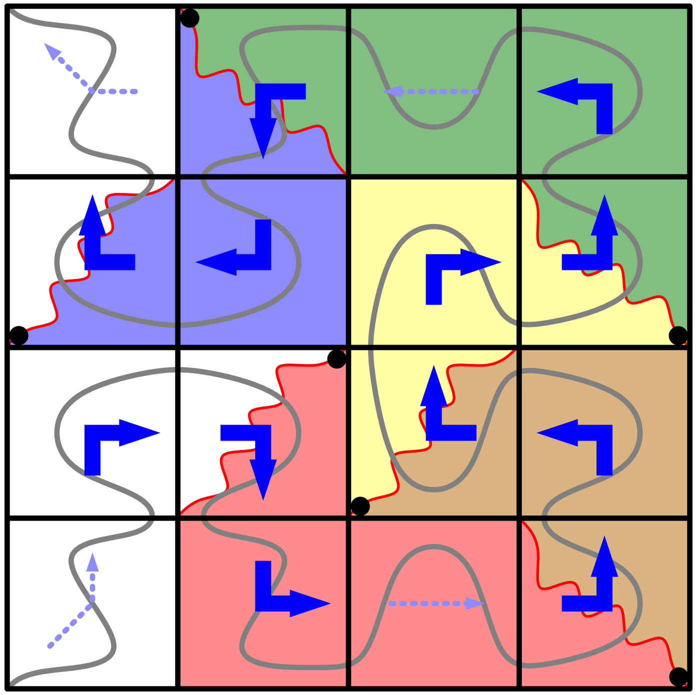

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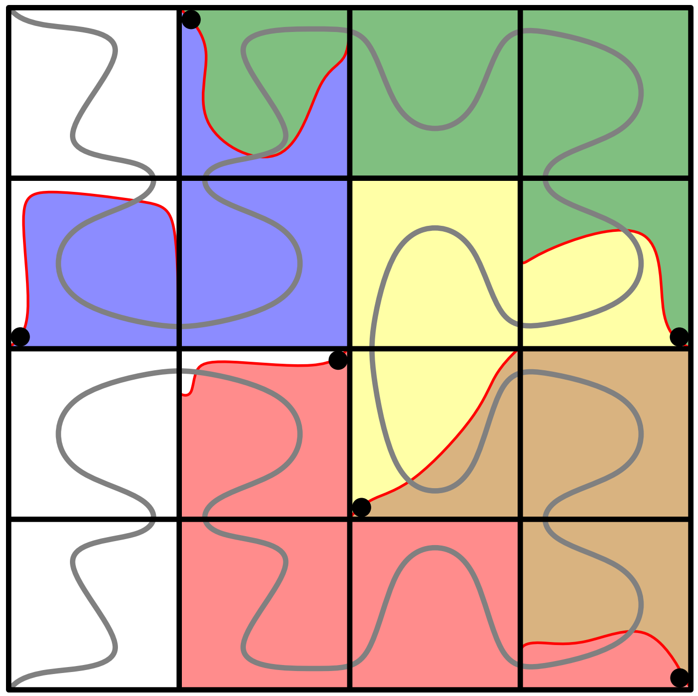

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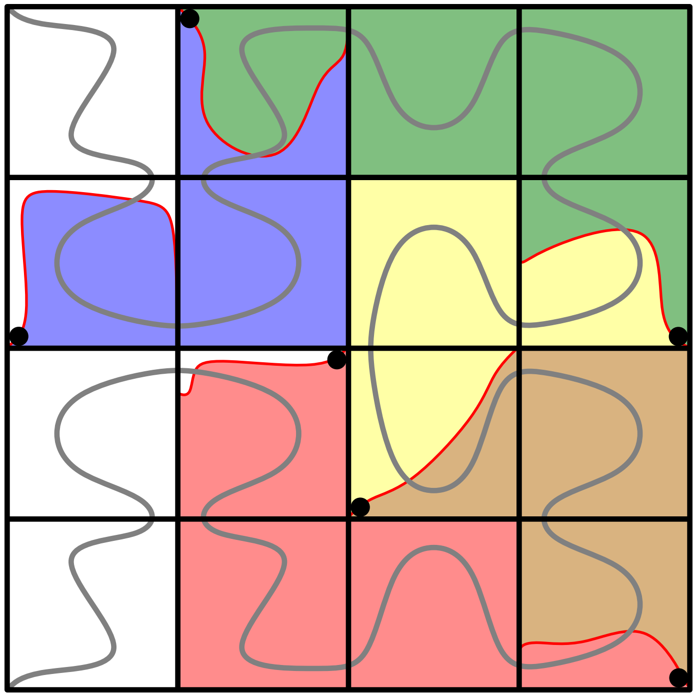

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- Grid size $n \times n$

$$\sum_{\text{consecutive } p,q} f_q - f_p \leqslant 1$$

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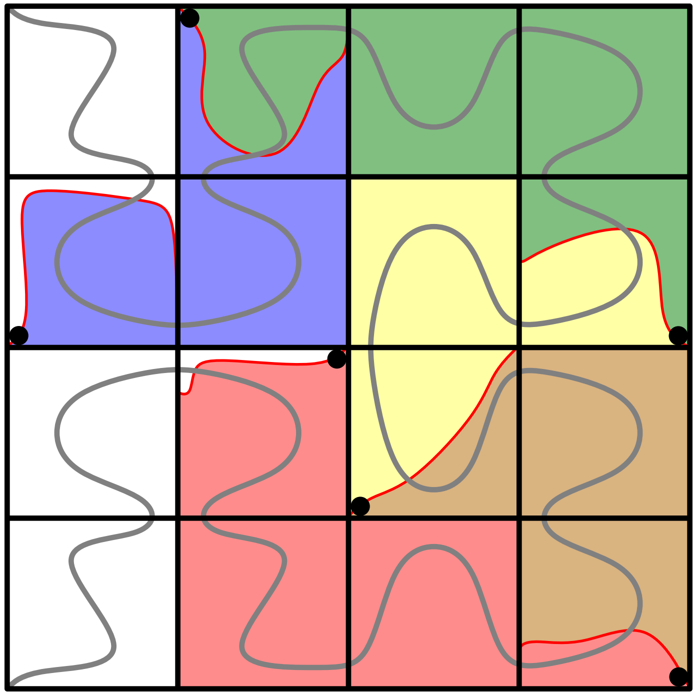

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$$\exists p, q : f_q - f_p = O\left(\frac{1}{n}\right)$$

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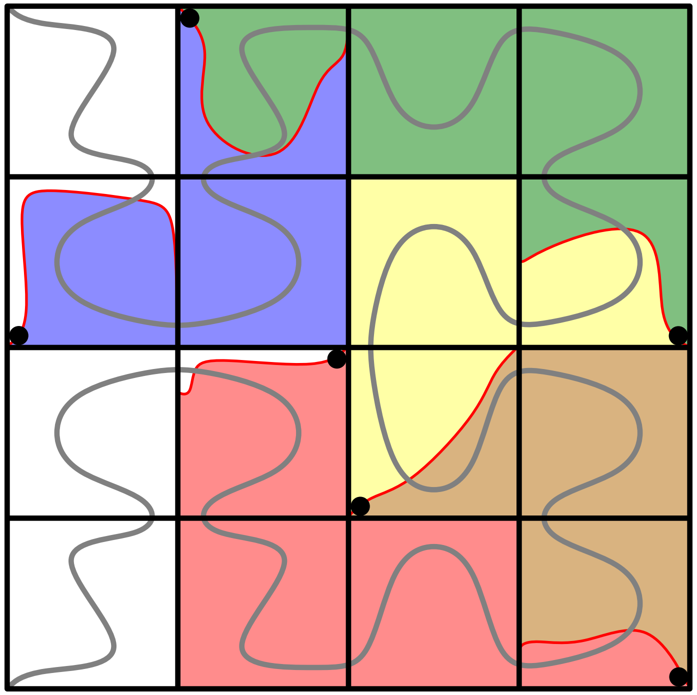

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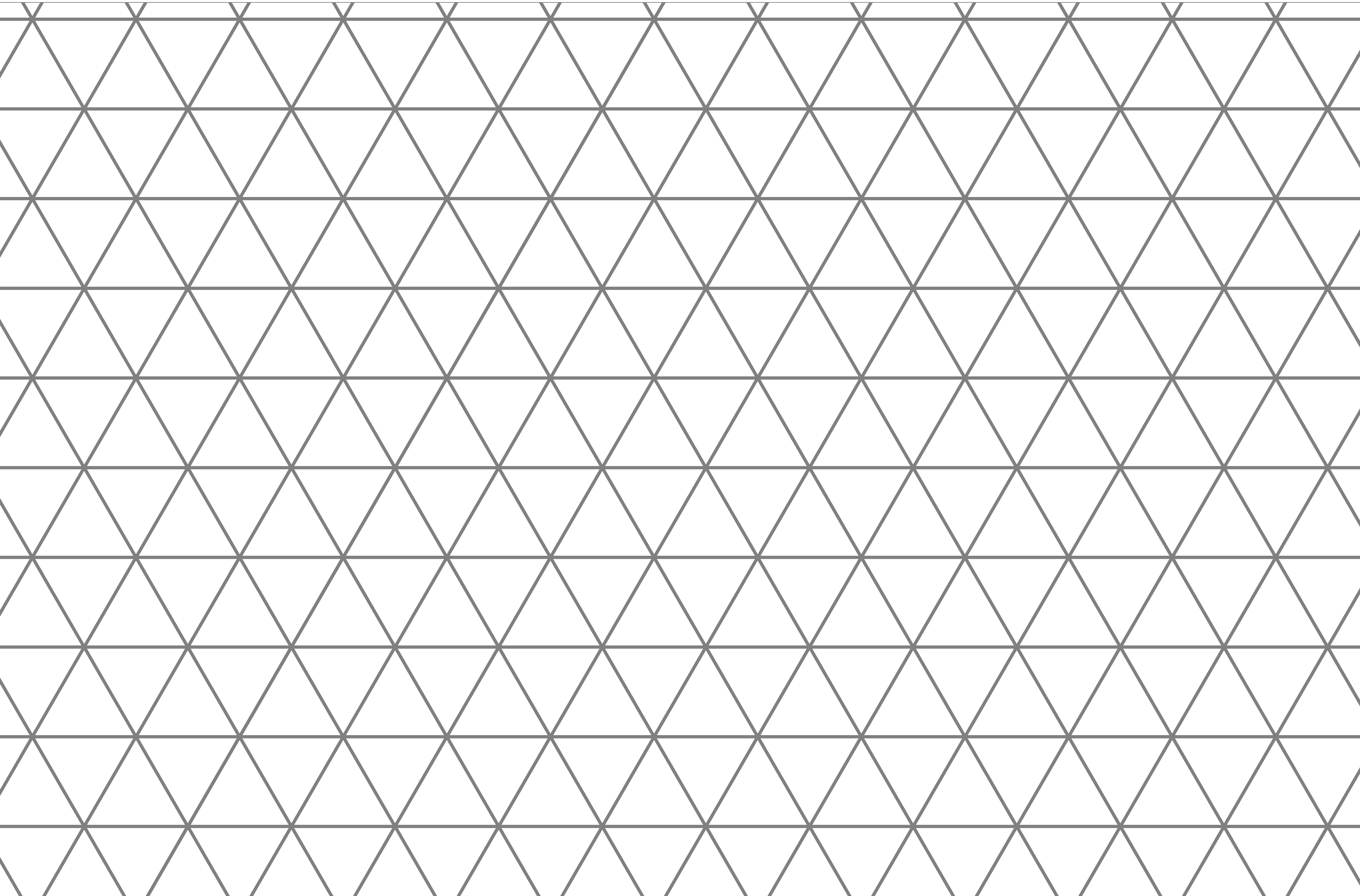
$$n \to \infty \Rightarrow f_q - f_p < C$$

Non-rectangular tilings?



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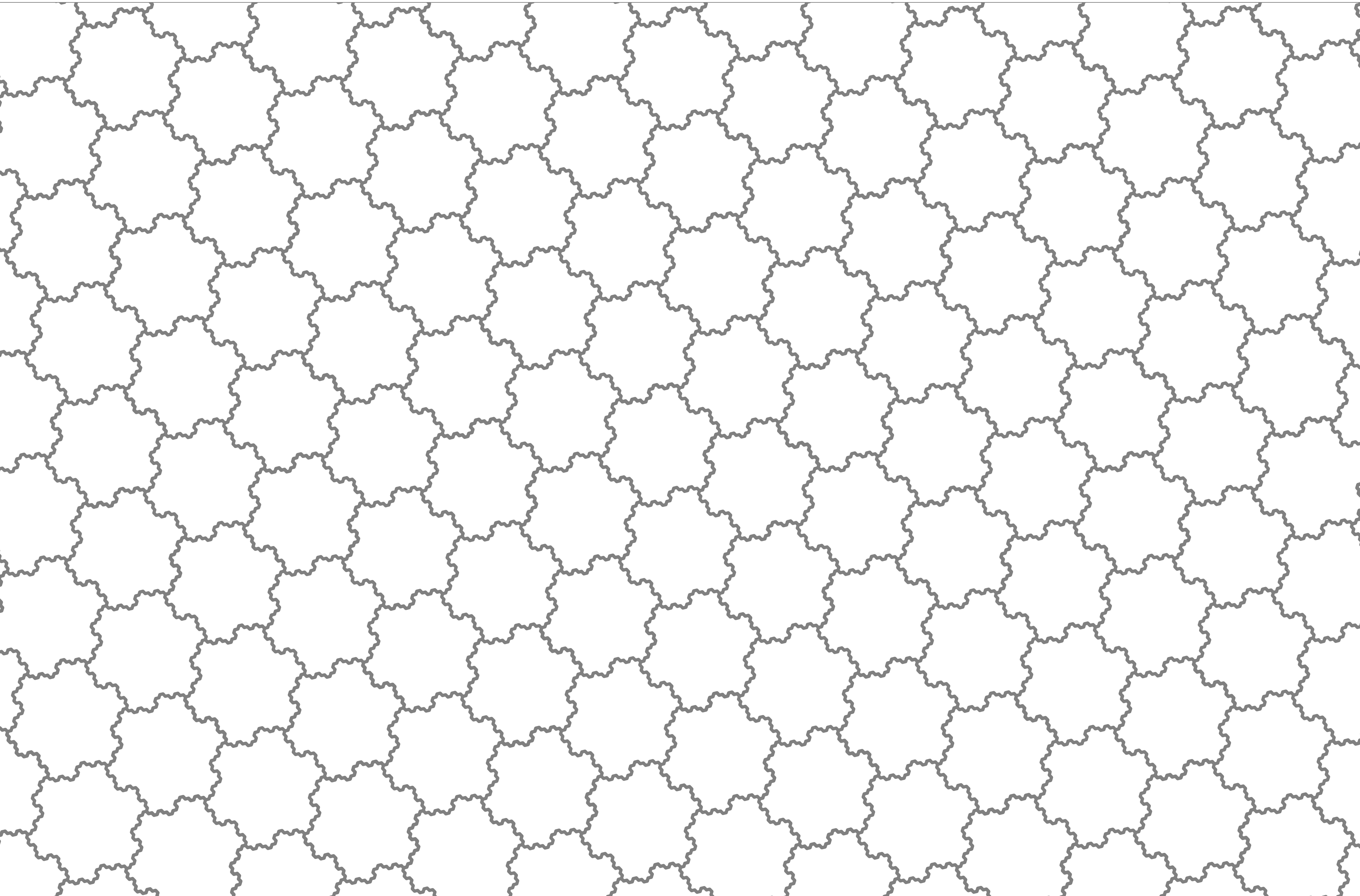

Further lower-bound results

Trivial:

- Triangle-based curves: $\mathrm{WBA} \geqslant 2$.

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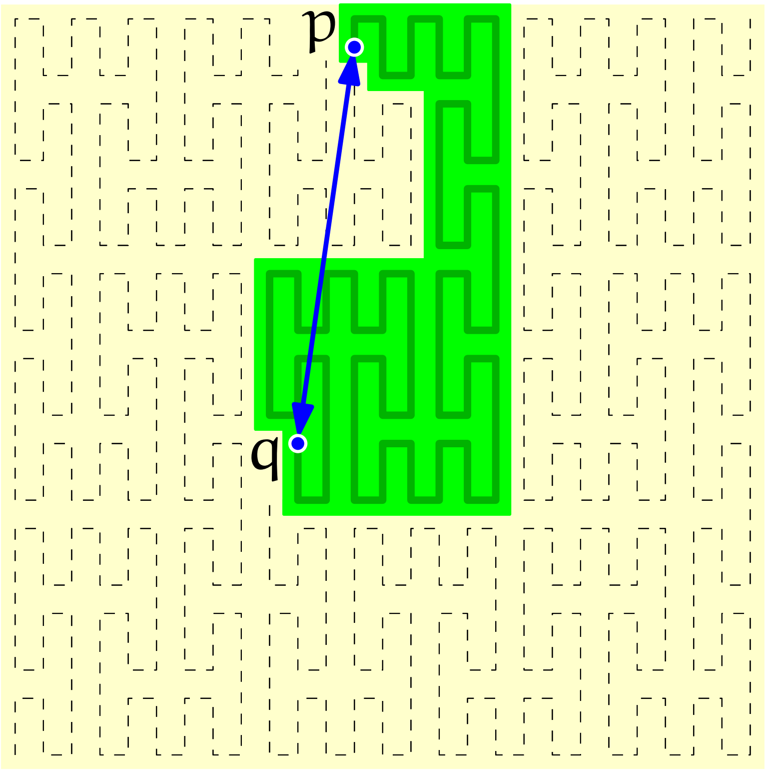

For a curve section of fixed size, how far can the endpoints be apart?

Further lower-bound results

Trivial:

- Triangle-based curves: $\mathrm{WBA} \geqslant 2$.

Using our proof technique:

- $\mathrm{WL}_2 \geqslant \mathrm{WL}_\infty \geqslant 4$.
  (Previously, for square-based curves: $\geqslant 3\,{}^1\!/_2$ in general, $\geqslant 4$ for cyclic curves.)

Using a technique from Niedermeier et al. [2002]:

- Triangle-based curves: $\mathrm{WL}_2 \geqslant 4$.

Computational results

- Approximation algorithm exploits recursive nature of curves (with a catch).

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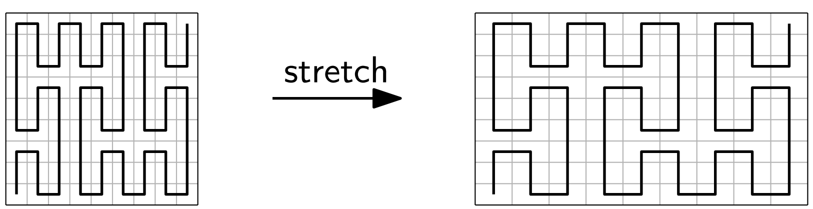
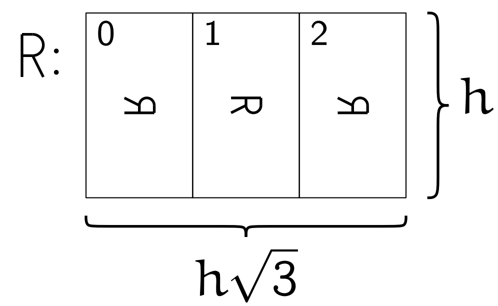


Peano's curve — "Balanced Peano"

Defining rule:



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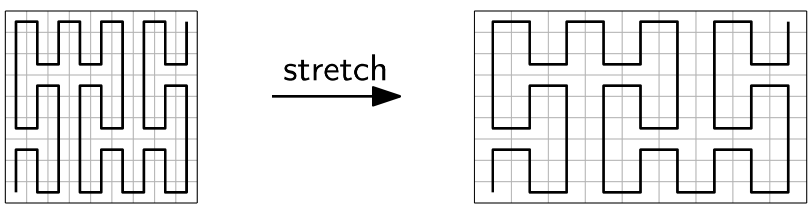
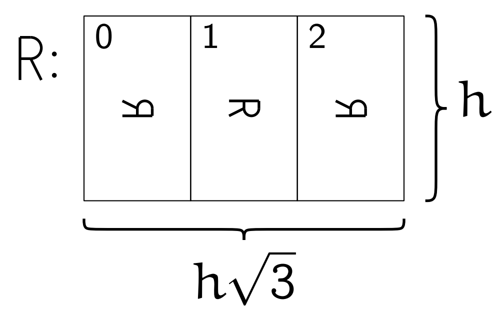
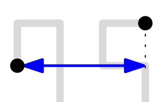
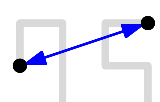
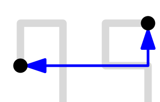
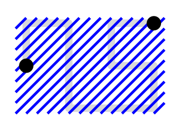
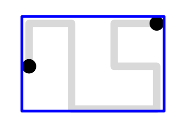
  WL-measures of curves by Peano, Hilbert, Sierpińsky–Knopp, etc.
- New curve with good locality:



Peano's curve   "Balanced Peano"

Defining rule:



	WL_∞	WL_2	WL_1	WBA	WBP
					
Peano	8	8	$10\,^2/_3$	2.000	2.722
Balanced Peano	4.619	4.619	8.619	2.000	2.155
Sierpińsky–Knopp	4	4	8	3.000	3.000

Open questions

- Lower bounds $\mathrm{WL_p}$ and WBA for other classes of space-filling curves?
- Improve lower bound $\mathrm{WBP} \geqslant \mathrm{WBA} \geqslant 2$?
  Curve with $\mathrm{WBP} = 2$? (Balanced Peano: $\mathrm{WBP} = 2.155$.)
- Three-dimensional space-filling curves?